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Monotonicity of unitarily invariant norms

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Any matrix unitarily invariant norm gives rise to a symmetric gauge function of the singular values of its matrix argument, but the dependency on the singular values is not equally weighted among them in the sense that the norm may not increase with some of the singular values under sufficiently small increases while it always increases when some other singular values increase no matter how tiny the changes are. This paper introduces and characterizes (argument-)dependent classifications of unitarily invariant norms and in particular the class in which a unitarily invariant norm increases with each of the first k largest singular values.
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1. Introduction

Consider the $m \times n$ matrix space $\mathbb{C}^{m\times n}$ with complex entries, and assume, without loss of generality, $m \geq n$. A matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is called a *unitarily invariant norm* if [1,2,4]

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1. $\|UXV\| = \|X\|$ for all $X \in \mathbb{C}^{m\times n}$;
2. $\|X\| = \|X\|_2$, the spectral norm, for all $X \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(X) = 1$.

Most commonly used unitarily invariant norms are the Frobenius norm $\|X\|_{\mathrm{F}}$, the spectral norm (2-norm) $\|X\|_2$, and the Ky Fan k-norms $\|X\|_{(k)}$ for $1 \le k \le n$.

We call a real-valued function Φ on $\mathbb{R}^n$ a *symmetric gauge function* if

1. Φ is a vector norm on $\mathbb{R}^n$;
2. $\Phi(Px) = \Phi(x)$ for any $n \times n$ permutation matrix P;
3. $\Phi(|x|) = \Phi(x)$, where $|x|$ is x taking entrywise absolute value;
4. $\Phi(e_1) = 1$, where $e_1 = [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^n$ and the superscript "T" takes the transpose.

A well-known theorem of von Neumann [5] established a one–one correspondence between unitarily invariant norms $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ and symmetric gauge functions Φ on $\mathbb{R}^n$ via[3] (see [1, p. 91], [4, p. 78])

$$\|X\| = \Phi(\sigma_1, \sigma_2, \dots, \sigma_n),$$

where $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_n$ are the singular values of X. For the aforementioned unitarily invariant norms, we have

1. the Ky Fan k norms $(1 \le k \le n)$

$$\|X\|_{(k)} := \sum_{i=1}^{k} \sigma_i =: \Phi_{(k)}(\sigma_1, \sigma_2, \dots, \sigma_n),$$

   and in particular the spectral norm $\|X\|_2 = \|X\|_{(1)}$;
2. the Frobenius norm

$$\|X\|_{\mathrm{F}} := \left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2} =: \Phi_{\mathrm{F}}(\sigma_1, \sigma_2, \dots, \sigma_n).$$

It is interesting to notice that not all singular values play equal roles in determining a given unitarily invariant norm. For example, $\|X\|_2$ depends only on σ_1, the largest singular value of X, in the sense that $\|X\|_2$ remains the same under sufficiently tiny changes to any of the smaller singular values. On the other hand, $\|X\|_{\mathrm{F}}$ depends on and increases with all singular values. Along this line, this paper investigates the monotonicity properties of unitarily invariant norms with respect to the singular values of their argument.

[3] For $x = [\xi_i] \in \mathbb{R}^n$, we will write either $\Phi(x)$ or $\Phi(\xi_1, \xi_2, \dots, \xi_n)$ whichever is more convenient.

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