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On properties of Karlsson Hadamards and sets of mutually unbiased bases in dimension six

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ABSTRACT

The complete classification of all 6×6 complex Hadamard matrices is an open problem. The 3-parameter Karlsson family encapsulates all Hadamards that have been parametrised explicitly. We prove that such matrices satisfy a non-trivial constraint conjectured to hold for (almost) all 6×6 Hadamard matrices. Our result imposes additional conditions in the linear programming approach to the mutually unbiased bases problem recently proposed by Matolcsi et al. Unfortunately running the linear programs we were unable to conclude that a complete set of mutually unbiased bases cannot be constructed from Karlsson Hadamards alone.

1. Introduction

In prime-power dimensions there are several ingenious methods to construct a complete set of mutually unbiased bases (MUBs) making use of finite fields, the Heisenberg–Weyl group, generalised angular momentum operators, and identities from number theory. However, even for the smallest composite dimension $d = 6$, the existence of such a set remains an open problem (see [1] for a review). The long-standing conjecture

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(also open in the study of Lie algebras [2]) is that in non-prime-power dimensions complete sets do not exist. Distinguishing quantum systems based on their number theoretic properties would be an unusual feature not seen in Classical mechanics. The additional symmetry in dimensions such as $d = 3 \times 3$ would appear to enlarge the boundary of the set of quantum correlations (potentially) resulting in larger violations of Bell inequalities.

Two recent papers have made progress in proving the non-existence of a complete set of MUBs in dimension 6. The first regards the classification of all complex Hadamard matrices. A complete set of $d + 1$ MUBs is equivalent to a set of d complex Hadamard matrices plus the identity matrix. Thus finding all possible Hadamard matrices in dimension 6 would be a significant step towards solving the MUBs problem. Recently, Karlsson has found an extremely nice 3-parameter family [3]. Whilst this does not fully classify all 6×6 Hadamard matrices [4], it includes all explicitly parametrised families (except the isolated Spectral matrix [5]). Moreover, Karlsson's parametrisation using Modius transformations is succinct and has allowed us to perform the calculations here.

The second result we use is due to Matolcsi et al., who demonstrate that sets of complex Hadamard matrices can be expressed as elements from a compact abelian group [6,7]. Fourier analysis then allows us to re-write the problem using the dual group where existence (or not) corresponds to a *linear* program. This method was used to classify all complete sets of MUBs in dimensions $d = 2 \dots 5$ and obtain some partial results in $d = 6$ [7].

In this paper, we prove that Karlsson Hadamard matrices satisfy a previously unknown constraint motivated by the linear programming approach of Matolcsi et al. Feeding this new condition into the linear program, we attempt to prove that a complete set of MUBs cannot be constructed from Karlsson Hadamards alone. Unfortunately we were unsuccessful in this endeavour but believe this is a promising avenue of research as our result can be easily combined with other conditions. Previous attempts to prove non-existence have required extensive computational power [8,9]. The linear programming approach is very appealing from this perspective. Using sparse matrix methods, we had no problems running the program in MATLAB on a desktop PC.

2. Conditions on unbiased sets of complex Hadamard matrices

A $d \times d$ unitary matrix, H, is called a *complex Hadamard matrix* if its entries are all complex phases, $|H_{ij}|^2 = 1$, for $i, j = 1 \dots d$. Any set of $r + 1$ MUBs has an equivalent representation in terms of Hadamard matrices $\{I, H_1, \dots, H_r\}$, where I is the $d \times d$ identity element (see [10] for a summary of equivalences on sets of MUBs).

There is a zoo of Hadamard matrices in dimension six, easily accessible online [11]. At the time of writing, there where 10 families with various properties such as being symmetric or continuously deformed from the Fourier matrix. There is a 4-parameter family of Hadamards [4] and furthermore, it is known that 4 parameters are sufficient for a complete classification [12]. Unfortunately, the construction given in [4] is implicit making it difficult to work with.