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Parametrizing projections with selfadjoint operators

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Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal decomposition of a Hilbert space, with E_+, E_- the corresponding projections. Let A be a selfadjoint operator in $\mathcal{H}$ which is codiagonal with respect to this decomposition (i.e. $A(\mathcal{H}_+) \subset \mathcal{H}_-$ and $A(\mathcal{H}_-) \subset \mathcal{H}_+$). We consider three maps which assign a selfadjoint projection to A:

1. The graph map Γ: $\Gamma(A) =$ projection onto the graph of $A|_{\mathcal{H}_+}$.
2. The exponential map of the Grassmann manifold $\mathcal{P}$ of $\mathcal{H}$ (the space of selfadjoint projections in $\mathcal{H}$) at E_+: $\exp(A) = e^{i\frac{\pi}{2}A}E_+e^{-i\frac{\pi}{2}A}$.
3. The map p, called here the Davis' map, based on a result by Chandler Davis, characterizing the selfadjoint contractions which are the difference of two projections.

The ranges of these maps are studied and compared.
Using Davis' map, one can solve the following operator matrix completion problem: given a contraction $a : \mathcal{H}_- \to \mathcal{H}_+$, complete the matrix

$$\begin{pmatrix} * & a/2 \\ a^*/2 & * \end{pmatrix}$$

to a projection P, in order that $\|P - E_+\|$ is minimal.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of the Hilbert space $\mathcal{H}$, and denote by E_+ and E_- the projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. If X is a selfadjoint operator in $\mathcal{H}$ which is codiagonal with respect to the decomposition (i.e. $X(\mathcal{H}_+) \subset \mathcal{H}_-$ and $X(\mathcal{H}_-) \subset \mathcal{H}_+$) and $x = X|_{\mathcal{H}_+}$, then $G_x = \{\xi \oplus x\xi : \xi \in \mathcal{H}_+\}$ is a closed subspace of $\mathcal{H}$. This is the usual way to obtain local charts for $\mathcal{P}$. Note that the trivial operator corresponds to the subspace $\mathcal{H}_+$. By means of the one-to-one map $\mathcal{S} \mapsto P_{\mathcal{S}}$ (= orthogonal projection onto $\mathcal{S}$), we identify $\mathcal{P}$ with the set of orthogonal projections in $\mathcal{H}$.

The graph map is the mapping which sends X to the orthogonal projection onto G_x.

The tangent space of the manifold $\mathcal{P}$ at E_+ (or $\mathcal{H}_+$) identifies with the space of selfadjoint codiagonal operators [7]. Thus the exponential map of $\mathcal{P}$ (from $T\mathcal{P}$ to $\mathcal{P}$) can also be regarded as a map from codiagonal selfadjoint operators to orthogonal projections.

The third map is obtained from a result by Chandler Davis [8], and will be defined below. Let us fix some notation.

Let J be the symmetry in $\mathcal{H}$ given by the decomposition of $\mathcal{H}$. J is a selfadjoint unitary operator ($J^* = J = J^{-1}$) whose spectral spaces are $\mathcal{H}_+$ and $\mathcal{H}_-$:

$$\mathcal{H}_+ = \{\xi \in \mathcal{H} : J\xi = \xi\} \quad \text{and} \quad \mathcal{H}_- = \{\xi \in \mathcal{H} : J\xi = -\xi\}.$$

The corresponding projections are

$$E_+ = \frac{1}{2}(J+1) \quad \text{and} \quad E_- = \frac{1}{2}(1-J).$$

Let $\mathcal{B}_J$ be the space of selfadjoint operators which are codiagonal. A simple calculation shows that X is codiagonal if and only if it anti-commutes with J: $XJ = -JX$.

Let $\mathcal{D}_J$ be the unit ball of $\mathcal{B}_J$.

In Theorem 6.1 in [8] Chandler Davis proved that a contraction A which anti-commutes with the symmetry J (i.e. $A \in \mathcal{D}_J$) is a difference of projections: $A = P_J - Q_J$ (with explicit formulas for P_J and Q_J). We define Davis' map as $p_J(A) = P_J$.

We shall consider the following subsets of $\mathcal{D}_J$. Let

$$\mathcal{D}_J^p = \mathcal{D}_J \cap \mathcal{B}_p(\mathcal{H}),$$

where $\mathcal{B}_p(\mathcal{H})$ is the ideal of p-Schatten operators ($1 \le p \le \infty$), with $\mathcal{B}_\infty(\mathcal{H})$ the ideal of compact operators. In [4] and [1] the notion of Fredholm pairs of projections was defined. Namely, a pair (P, Q) of projections is a Fredholm pair if

$$QP|_{R(P)} : R(P) \to R(Q)$$