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Generalized counting constraint satisfaction problems with determinantal circuits

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Generalized counting constraint satisfaction problems include Holant problems with planarity restrictions; polynomial-time algorithms for such problems include matchgates and matchcircuits, which are based on Pfaffians. In particular, they use gates which are expressible in terms of a vector of sub-Pfaffians of a skew-symmetric matrix. We introduce a new type of circuit based instead on determinants, with seemingly different expressive power. In these *determinantal circuits*, a gate is represented by the vector of all minors of an arbitrary matrix. Determinantal circuits permit a different class of gates. Applications of these circuits include proofs of theorems from algebraic graph theory including the Chung–Langlands formula for the number of rooted spanning forests of a graph and computing Tutte polynomials of certain matroids. They also give a strategy for simulating quantum circuits with closed timelike curves. Monoidal category theory provides a useful language for discussing such counting problems, turning combinatorial restrictions into categorical properties. We introduce the counting problem in monoidal categories and count-preserving functors as a way to study FP subclasses of problems in settings which are generally #P-hard. Using this machinery we show that, surprisingly, determinantal circuits can be simulated by Pfaffian circuits at quadratic cost.

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1. Introduction

Let $\mathrm{Vect}_{\mathbb{C}}$ be the category of finite-dimensional vector spaces and linear transformations over the base field $\mathbb{C}$. A string diagram [11] in $\mathrm{Vect}_{\mathbb{C}}$ is a *tensor* (*contraction*) *network*. Fixing such a diagram, the problem of computing the morphism represented is the *tensor contraction problem*, which is in general #P-hard (examples include weighted counting constraint satisfaction problems [7]).

We study complex-valued tensor contraction problems in subcategories of $\mathrm{Vect}_{\mathbb{C}}$ by considering them as diagrams in a monoidal category. For a survey of the rich diagrammatic languages that can be specified similarly see [23] and the references therein. By a *circuit* we mean a combinatorial counting problem expressed as a string diagram in a monoidal subcategory of $\mathrm{Vect}_{\mathbb{C}}$ (that is, a tensor contraction network). Such diagrams generalize weighted constraint satisfaction problems and Boolean circuits (such as by requiring planarity), and are often related to existing description languages. Subcategories of $\mathrm{Vect}_{\mathbb{C}}$ can faithfully represent Boolean [14] and quantum circuits [2], counting constraint satisfaction problems, and many other problems [9].

Suppose we have a problem $\mathcal{L}$; a common example are counting constraint satisfaction problems [6], perhaps with some restrictions such as planarity. Such a problem can be described by the data of a monoidal word (see e.g. [13, Chapter 12]) and a *interpretation* [23] map $i : \mathcal{L} \to \mathcal{C}$ that assigns values to primitive terms in the word. Then determining which morphism is obtained is a tensor contraction problem in some monoidal category $\mathcal{C}$.

From the point of view of complexity theory, we are interested in the class FP which is comprised of the functions $\{0,1\}^* \to \mathbb{N}$ computable by a deterministic polynomial-time Turing machine (see e.g. [1, p. 344]). A second functor $\mathbf{h} : \mathcal{C} \to \mathcal{S}$ from a category $\mathcal{C}$ in which the contraction problem (Problem 2.1) is in FP and a subcategory $\mathcal{S}$ of $\mathrm{Vect}_{\mathbb{C}}$ that preserves the solution to the FP problem serves to characterize the problems which can be solved in polynomial time according to a particular contraction scheme.

The motivation of this paper comes from holographic algorithms [24] and our attempts to generalize it and give it a uniform language. This and related schemes work by exploiting some combinatorial identity or *kernel* relating an exponential sum (corresponding to performing the tensor contraction by a naïve algorithm) and a polynomial time operation that yields the same result. They can be viewed as a complementary alternative method to geometric complexity theory [22] in the study of which counting problems (such as computing a permanent) may be embedded in a determinant computation at polynomial cost.

We formulate a class of circuits based on determinants and show that the corresponding tensor contraction problem is solvable in polynomial time. The existence of such a class was conjectured in [15]. A circuit class based on Pfaffians of minors had already been given [18] and the formula for the number of rooted spanning forests of a graph [8] hinted at the kernel to use for determinantal circuits. Indeed, we can recover the theorem using determinantal circuits.