Linear Algebra and its Applications 466 (2015) 382–400

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Each symplectic matrix is a product of four symplectic involutions

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ARTICLE INFO

Article history:
Received 8 July 2014
Accepted 10 October 2014
Available online 29 October 2014
Submitted by V.V. Sergeichuk

MSC:
15A21
15A23

Keywords:
Symplectic matrices
Involutions
Coninvolutions
Square zero matrices
Hamiltonian matrices

ABSTRACT

Gustafson, Halmos, and Radjavi in 1973 proved that each matrix A with $\det A = \pm 1$ is a product of four involutions. We prove that these involutions can be taken to be symplectic if A is symplectic (every symplectic matrix has unit determinant). Using this result we give an alternative proof of Laffey's theorem that every nonsingular even size matrix is a product of skew symmetric matrices. Ballantine in 1978 proved that each matrix A with $|\det A| = 1$ is a product of four coninvolutions. We prove that these coninvolutions can be taken to be symplectic if A is symplectic. We also prove that each Hamiltonian matrix is a sum of two square zero Hamiltonian matrices.

1. Introduction

We show that *each symplectic matrix of size greater than two is a product of four symplectic involutions* ($A^2 = I$) *or four symplectic coninvolutions* ($A\overline{A} = I$) (see

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http://dx.doi.org/10.1016/j.laa.2014.10.016

Theorems 27 and 32). Recall that a nonsingular matrix $A \in \mathbb{C}^{2n\times 2n}$ is called *symplectic* if $J^{-1}A^TJ = A^{-1}$, where

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

A symplectic matrix A preserves the bilinear form $\langle x, y\rangle_J := x^TJy$, that is, $\langle Ax, Ay\rangle_J = \langle x, y\rangle_J$ for all $x, y \in \mathbb{C}^{2n\times 2n}$.

A lot of work has been done on the products of involutions and structured involutions. Wonenburger [33] (see also [5,10,17]) used the theory of invariant factors to show that a matrix is a product of two involutions if and only if it is nonsingular and similar to its inverse. Wonenburger [32,33] also proved that each orthogonal matrix is a product of two orthogonal involutions, and that each symplectic matrix is a product of two *skew symplectic* ($J^{-1}A^TJ = -A^{-1}$) involutions. Halmos and Kakutani [16] proved that each unitary operator on an infinite dimensional complex Hilbert space is a product of four involutions. Radjavi [29] obtained a similar result for the group of all unitary operators with determinant ± 1 on finite dimensional complex Hilbert spaces. He also proved that each matrix of size $n > 1$ with determinant ± 1 is a product of $2n - 1$ *simple involutions* ($A^2 = I$ and $\operatorname{rank}(A - I) = 1$), see [30]. By Liu [25], if $T \in \mathbb{C}^{n\times n}$, $\det T = \pm 1$ and $\dim(\ker(T - zI)) \le \lfloor n/2 \rfloor$ for each complex number z, then T is a product of three involutions. He also proved that if T is a product of three involutions, then $\dim(\ker(T - zI)) \le \lfloor 3n/4 \rfloor$ for each complex number z such that $z^4 \ne 1$. Gustafson, Halmos, and Radjavi [15] used the rational canonical form of matrices to prove that each matrix with determinant ± 1 is a product of four involutions.

The product of coninvolutions has received much less attention. By Ballantine [6], a matrix T is a product of coninvolutions if and only if $|\det T| = 1$. Abara, Merino, and Paras [1] proved that T is a product of two coninvolutions if and only if T is similar to $\overline{T}^{-1}$. Coninvolutions are also studied because of their close relation to the theory of consimilarity and condiagonalizability, see [19,22].

Lin, Mehrmann, and Xu [24] used the theory of Kronecker canonical forms and symplectic matrix pencils to give a canonical form of *conjugate symplectic* matrices ($J^{-1}A^*J = A^{-1}$) with respect to conjugate symplectic similarity. Horn and Merino [18] proved that two symplectic matrices are similar if and only if they are symplectically similar. We use this to give a *canonical form of symplectic matrices with respect to symplectic similarity* (see Lemma 5).

Denote by $\mathcal{SI}_{2n}(m)$ the set of all products of m symplectic involutions. We show that

$$\mathcal{SI}_{2n}(1) \subset \mathcal{SI}_{2n}(2) \subset \mathcal{SI}_{2n}(3) \subset \mathcal{SI}_{2n}(4),$$

and that these inclusions are strict. We prove equivalent conditions for a symplectic matrix to be a product of two symplectic involutions, one of which is that its number of Jordan blocks of size k corresponding to each eigenvalue λ is even, see Theorem 8.