Linear Algebra and its Applications 466 (2015) 409–420

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Cayley sum color and anti-circulant graphs

M. Amooshahi, B. Taeri *

Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran

ARTICLE INFO

Article history:
Received 15 March 2014
Accepted 20 October 2014
Available online 31 October 2014
Submitted by D. Stevanovic

MSC:
05C25
05C50
15A18

Keywords:
Cayley sum color graph
Circulant matrix
Anti-circulant matrix

ABSTRACT

Let G be a finite group and $\alpha : G \to \mathbb{R}$ be a real-valued function on G. The Cayley sum color graph $\mathrm{Cay}^+(G,\alpha)$ is a complete directed graph with vertex set G where each arc $(x,y) \in G \times G$ is associated with a color $\alpha(xy)$. If α is the characteristic function on a subset S of G, then the Cayley sum graph $\mathrm{Cay}^+(G,S)$ is obtained. The anti-circulant matrix associated to a vector v is an $n \times n$ matrix whose rows are given by iterations of the anti-shift operator acting on v. We note that a graph is a Cayley sum graph of a cyclic group if and only if it is an anti-circulant graph, a graph whose adjacency matrix is anti-circulant. In this paper, we obtain some results on the isomorphisms, connectivity and vertex transitivity about anti-circulant graphs. We find the spectrum of Cayley sum color graphs of abelian groups and as a result we compute the spectrum of real anti-circulant matrices.

1. Introduction

Let G be a finite group and $\alpha : G \to \mathbb{R}$ be a real-valued function on G. The Cayley sum color graph $\mathrm{Cay}^+(G,\alpha)$ is the complete directed graph with vertex set G where each arc $(x,y) \in G \times G$ is associated with a color $\alpha(xy)$. If α is the characteristic function of a subset $S \subseteq G$, that is,

* Corresponding author.
E-mail addresses: m.amooshahi@math.iut.ac.ir (M. Amooshahi), b.taeri@cc.iut.ac.ir (B. Taeri).

http://dx.doi.org/10.1016/j.laa.2014.10.031
0024-3795/

$$\alpha(g) = \begin{cases} 1 & \text{if } g \in S \\ 0 & \text{if } g \notin S, \end{cases}$$

then we use $\mathrm{Cay}^+(G, S)$ instead of $\mathrm{Cay}^+(G, \alpha)$ and call it the Cayley sum digraph of G with respect to S. We regard $\mathrm{Cay}^+(G, S)$ as a digraph formed by deleting the arcs of color 0 from the color graph.

The adjacency matrix A of a Cayley sum color graph $\mathrm{Cay}^+(G, \alpha)$ is a $|G| \times |G|$ matrix whose row-g column-h entry a_{gh} is $\alpha(gh)$ for all $g, h \in G$. In a Cayley sum (di)graph $\mathrm{Cay}^+(G, S)$, there is an (arc) edge from g to h if and only if $a_{gh} = \alpha(gh) = 1$, equivalently $h = g^{-1}s$, for some $s \in S$.

Note that if S is a normal subset of G, i.e. for every $g \in G$, $g^{-1}Sg = S$, then $\mathrm{Cay}^+(G, S)$ is an undirected graph. Also, if there exists $g \in G$ such that $g^2 \in S$, then the edge $\{g, g\}$ is a semi-edge; a semi-edge is an edge with one endpoint. Unlike a loop, a semi-edge contributes just one to both the valency of its endpoint and the corresponding diagonal entry of the adjacency matrix. With this convention, $\mathrm{Cay}^+(G, S)$ is a regular graph of degree $|S|$, as the set of adjacent vertices of g is $\{g^{-1}s \mid s \in S\}$. In the case that G is a finite abelian group, Cayley sum graphs are known in the literature under names *addition Cayley graphs* [8,10,11], *addition graphs* [3] and *sum graphs* [4].

The study of Cayley sum graphs has been the object of some papers, for example, independence number [2], hamiltonicity [3,10]. Also, the results of a paper by Chung concern expander properties and the diameter [4], Green studied the clique number in [7] and Grynkiewicz et al. worked on connectivity of Cayley sum graphs [8]. To some extent, this situation may be explained by the fact that Cayley sum graphs are rather difficult to study. For example, it is uncomplicated to prove that any connected Cayley graph on a finite abelian group with at least three elements is hamiltonian; however, apart from the results of [3], nothing seems to be known on hamiltonicity of addition Cayley graphs on finite abelian groups. Also, the connectivity of a Cayley graph on a finite abelian group is easy to determine, while determining the connectivity of a Cayley sum graph is a non-trivial problem, see [8].

Recall that a graph is called *circulant* if it has a circulant adjacency matrix. Let us recall the definition of circulant matrices. Fix a positive integer $n \geq 2$, and let $v = (c_0, c_1, \dots, c_{n-1})$ be a vector in $\mathbb{C}^n$. Define the shift operator $T : \mathbb{C}^n \to \mathbb{C}^n$ given by $T(c_0, c_1, \dots, c_{n-1}) = (c_{n-1}, c_0, \dots, c_{n-2})$. The circulant matrix $\mathrm{circ}(v)$, associated to the vector v is the $n \times n$ matrix whose rows are given by iterations of the shift operator T acting on v, i.e. its k-th row is $T^{k-1}v$, $k = 1, 2, \dots, n$:

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$