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A geometric description of the maximal monoids of some matrix semigroups

Raúl E. González-Torres

CINVESTAV – Unidad Guadalajara, Mexico

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The maximal monoids of the form $F\mathbf{S}F$ are studied, where F is a nonnegative idempotent matrix and $\mathbf{S}$ is one of the following matrix semigroups: $\mathbf{N}_n$, the nonnegative square matrices, $\mathbf{St}_n$, the stochastic matrices, and $\mathbf{D}_n$, the doubly stochastic matrices. For the cases of $\mathbf{N}_n$ and $\mathbf{St}_n$, it is shown that these maximal monoids are affinely isomorphic to the full semigroups of lower order, and for $F\mathbf{D}_nF$ that it is a compact affine semigroup with zero, here called the *core* of a primitive idempotent.

1. Introduction

In this paper we give an alternative description of the maximal monoids of nonnegative, stochastic and doubly stochastic matrices to the one presented by D.J. Hartfiel and C. Maxson in [14], and state in a different context the results about maximal groups of stochastic and doubly stochastic matrices obtained by H.K. Farahat [9] and S. Schwarz [16].

E-mail address: Raul.Gonzalez@gdl.cinvestav.mx.

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All this study is placed in the context of compact affine semigroups, thus intending to exploit the interplay that exists between topological semigroups and the geometries of affine spaces and convex polytopes.

In the remainder of this section we state the basic facts and terminology.

A (compact) semigroup $\mathbf{S}$ on a real locally convex topological vector space is said to be (*compact*) *affine* if $\mathbf{S}$ is a (compact) convex set and its binary operation respects convex combinations; that is, if for every three elements x, y, z of $\mathbf{S}$ and every real number λ in the unit interval $[0,1]$ it holds that $(\lambda x + (1-\lambda)y)z = \lambda(xz) + (1-\lambda)(yz)$ and $z(\lambda x + (1-\lambda)y) = \lambda(zx) + (1-\lambda)(zy)$. For the basic concepts regarding this type of semigroups we rely on the foundational work of H. Cohen and H.S. Collins [7]; for the concepts used here related to topological semigroups in general, please refer to J.H. Carruth et al. [4].

Let $\mathbf{S}$ and $\mathbf{T}$ be (compact) affine semigroups. An *affine homomorphism* from $\mathbf{S}$ to $\mathbf{T}$ is a semigroup homomorphism $\varphi : \mathbf{S} \longrightarrow \mathbf{T}$ that respects convex combinations of elements; that is, for every pair of elements x, y of $\mathbf{S}$ and every scalar λ in the unit interval $[0,1]$ it holds that $\varphi(\lambda x + (1-\lambda)y) = \lambda\varphi(x) + (1-\lambda)\varphi(y)$. When φ is also a bijective function, it is called an *affine isomorphism* between $\mathbf{S}$ and $\mathbf{T}$, and these semigroups are referred to as *affinely isomorphic*.

Let $\mathbf{M}_n$ denote the algebra of real square matrices of order n. Denote the *rank* of a matrix A of $\mathbf{M}_n$ by $r(A)$. The ith element of the standard orthonormal basis of $\mathbf{R}^n$ is here denoted by $\mathbf{e}_i^n$, where the subindex i also indicates the entry which is equal to 1, for $i = 1, 2, ..., n$. The symbol $\mathbf{u}^n$ is used to denote the column vector in $\mathbf{R}^n$ all of whose entries are equal to 1. In both instances, the superscript n will be omitted if there appears to be no place for confusion. A subsemigroup of $\mathbf{M}_n$ that has been extensively studied is $\mathbf{N}_n$, the semigroup of *nonnegative matrices* of order n [1]. Geometrically, $\mathbf{N}_n$ is a convex cone of dimension n^2 with vertex at the zero matrix Θ_n and whose edges are the nonnegative axes spanned by the order n matrices U_{ij} all of whose entries are equal to zero, except for a_{ij} which is equal to 1, for each $i, j = 1, 2, ..., n$.

The semigroup of *stochastic matrices* of order n, here denoted as $\mathbf{St}_n$, is the collection of all nonnegative matrices A for which $A\mathbf{u} = \mathbf{u}$; that is, the sum of the entries on each row of A is equal to 1. The semigroup of *doubly stochastic matrices* of order n, denoted as $\mathbf{D}_n$, is the collection of all stochastic matrices D for which $D^{\mathbf{T}}\mathbf{u} = \mathbf{u}$; that is, the sum of the entries on each row and column is equal to 1. The set $\mathbf{St}_n$ constitutes a compact affine subsemigroup with identity I_n of the multiplicative semigroup $\mathbf{N}_n$. It is also a convex polytope of dimension $n(n-1)$ whose vertices are all the stochastic matrices having 0 or 1 in any of their entries (n^n vertices in total). The set $\mathbf{D}_n$ constitutes a compact affine subsemigroup of $\mathbf{St}_n$ with identity I_n. It is also a convex polytope (the well-known *assignment, or Birkhoff, polytope* [17]) of dimension $(n-1)^2$, whose vertices constitute the permutation matrices of order n; that is, the vertices of $\mathbf{St}_n$ that are nonsingular matrices. These are known to form a group (here denoted as $\mathbf{P}_n$) isomorphic to the symmetric group on n letters S_n, and hence there are exactly $n!$ of them.