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On the smallest eigenvalues of the line graphs of some trees

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In this paper, we study the characteristic polynomials of the line graphs of generalized Bethe trees. We give an infinite family of such graphs sharing the same smallest eigenvalue. Our family generalizes the family of coronas of complete graphs discovered by Cvetković and Stevanović.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. By an eigenvalue of a graph we mean an eigenvalue of its adjacency matrix. It is well known that graphs whose smallest eigenvalue is greater than -2 are the line graphs of trees, or the line graphs of a unicyclic graph with an odd cycle, certain generalized line graphs of trees, or exceptional graphs arising from the root system E_8 (see [5]).

In this paper, we focus on the line graphs of trees and study the smallest eigenvalue of a particular type of such graphs. Our research is motivated by a question raised by Cvetković and Stevanović. In [4], it is shown that the sequence $\{\lambda_{\min}(K_n \otimes K_q)\}_{n=1}^{\infty}$ is constant for a fixed integer q, where $\otimes$ denotes the corona of graphs (see page 51 of [3] for a definition of corona and see also [6] for its generalization), and $\lambda_{\min}$ denotes the smallest eigenvalue. Cvetković and Stevanović raised the following question:

Question 1. Do there exist other sequences of the line graphs of trees whose smallest eigenvalues are constant?

In this paper, we give an answer for this question by giving a larger family of graph sequences of the line graphs of trees which have a constant smallest eigenvalue (Corollary 13).

For positive integers $d_1 = 1$, $d_2 \geq 2, \dots, d_{k-1} \geq 2$, $d_k \geq 1$, we define a tree $B(d_1, \dots, d_k)$ to be a rooted tree with k levels in which every vertex at level j has degree d_{k-j+1}. Note that $K_n \otimes K_q$ is isomorphic to the line graph $L(B(1, q, n))$ of the tree $B(1, q, n)$. The tree $B(d_1, \dots, d_k)$ is called a *generalized Bethe tree* (see [8,9]).

In the next section, we determine the characteristic polynomial of the line graph $L(B(d_1, \dots, d_k))$ of the tree $B(d_1, \dots, d_k)$ in a factored form, thereby showing that the smallest eigenvalue of $L(B(d_1, \dots, d_k))$ is independent of d_k (Theorem 9). We also show that the smallest eigenvalue of $L(B(d_1, \dots, d_k))$ has multiplicity $d_k - 1$, and is a zero of a polynomial of degree $k - 1$ (Theorem 12). The characteristic polynomial of $L(B(d_1, \dots, d_k))$ has also been determined by Rojo and Jiménez [8] using a different method, but our result gives more concrete information about the smallest eigenvalue.

2. The characteristic polynomial

We denote by $\chi_G(\lambda)$ the characteristic polynomial of the adjacency matrix $A(G)$ of a graph G, that is, $\chi_G(\lambda) = \det(\lambda I - A(G))$.

Let G and H be rooted graphs with roots u and v, respectively. We denote by $G \cdot H$ the graph obtained from G and H by identifying the vertices u and v.

Lemma 2. *(See Schwenk [10, Corollary 2b].) Let G and H be rooted graphs with roots u and v, respectively. Then*

$$\chi_{G \cdot H}(\lambda) = \chi_{G-u}(\lambda)\chi_H(\lambda) + \chi_G(\lambda)\chi_{H-v}(\lambda) - \lambda\chi_{G-u}(\lambda)\chi_{H-v}(\lambda).$$