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On the inclusion matrix $W_{23}(v)$

M.H. Ahmadi, N. Akhlaghinia, G.B. Khosrovshahi *, Ch. Maysoori

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

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For a v-set X, $W_{23}(v)$ is a $\binom{v}{2} \times \binom{v}{3}$ inclusion matrix where rows and columns are indexed by pairs and triples of X, respectively, and for row T and column K, $W_{23}(v)(T,K) = 1$ if $T \subseteq K$ and zero otherwise. In this paper, we classify the basis elements of the $\mathrm{null}_{\mathbb{Z}}(W_{23}(v))$, derived from the Gaussian elimination on $W_{23}(v)$ (called standard basis), into five classes. Then, we present a new algorithm to construct a $(2,3,v)$-halving for a feasible v, i.e. a nowhere zero $\mathrm{T}(2,3,v)$-trade.

1. Introduction

The inception of inclusion matrices in the literature goes back to early seventies of the last century when two fundamental papers appeared almost simultaneously in 1973 [11,20]. Although these matrices have many applications in design theory [3,12,16,19,20] and coding theory [6,17], only recently they have become a focal of interest for their own

* Corresponding author.
E-mail address: rezagbk@ipm.ir (G.B. Khosrovshahi).

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rights [2,5,7,8,10,15,18]. Although the subject of this paper is to study a special case of these matrices, namely $W_{23}(v)$, nevertheless some of the definitions and theorems are stated in a more general forms.

Integers t, k, and v with $0 \leq t \leq k \leq v$ are considered. Let X be a linearly ordered v-set, and

$$\binom{X}{i} := \{A \subseteq X : |A| = i\}, \quad 0 \leq i \leq v.$$

The *inclusion matrix* $W_{tk}(v)$ (for short W_{tk}) is defined to be a $\binom{v}{t}$ by $\binom{v}{k}$ $(0,1)$-matrix whose rows and columns are indexed by (and referred to) the members of $\binom{X}{t}$ and $\binom{X}{k}$, respectively, where

$$W_{tk}(v)(T,K) := \begin{cases} 1 & \text{if } T \subseteq K \\ 0 & \text{otherwise,} \end{cases} \quad T \in \binom{X}{t}, \; K \in \binom{X}{k}.$$

Let $\mathbf{1}$ be the all 1-vector, and λ be a non-negative integer. We call the following equation the fundamental equation of design theory,

$$W_{tk} f = \lambda \mathbf{1}. \tag{1}$$

- Every integral solution of Eq. (1) is called a *signed* t-(v,k,λ) *design*.
- For $\lambda > 0$, every non-negative integral solution of Eq. (1) is called a t-(v,k,λ) *design*.
- For $\lambda = 0$, every integral solution of Eq. (1) is called a $\mathrm{T}(t,k,v)$-*trade*.
- A $\mathrm{T}(t,k,v)$-trade with entries 1 or -1 is called a (t,k,v)-*halving*.

In this paper, we present a method for constructing a $(2,3,v)$-halving, which proves the following conjecture for $t = 2$ and $k = 3$.

Hartman's Conjecture. *For* $0 \leq i \leq t$, *there is a* (t,k,v)-*halving if and only if* $\binom{v-i}{k-i}$ *is even for* $i = 0, \ldots, t$.

For $t = 2$ and $k = 3$, Dehon [4] has proved that $\binom{X}{3}$ is halvable if and only if $v = 2 \pmod 4$. Furthermore, the conjecture has been verified by Ajoodani-Namini for $t = 2$ [1].

More specifically, in this paper, first we construct a basis for the null space of $W_{23}(v)$ coming from Gaussian elimination. Then by classifying the elements of this basis, we introduce a new algorithm for constructing a $(2,3,v)$-halving.

2. Notations and some necessary materials

In this section we provide some necessary information about trades, standard trades, standard basis, and starting blocks of trades.