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Some majorization inequalities in Euclidean Jordan algebras

J. Tao [a,*], Lingchen Kong [b], Ziyan Luo [c], Naihua Xiu [b]

[a] Department of Mathematics and Statistics, Loyola University Maryland, Baltimore, MD 21210, USA
[b] Department of Applied Mathematics, Beijing Jiaotong University, Beijing 100044, PR China
[c] State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, PR China

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ABSTRACT

In this paper, we prove various eigenvalue and trace inequalities of objects in the setting of simple Euclidean Jordan algebras via majorization techniques.

* Corresponding author.
E-mail addresses: jtao@loyola.edu (J. Tao), konglchen@126.com (L. Kong), zyluo@bjtu.edu.cn (Z. Luo), nhxiu@bjtu.edu.cn (N. Xiu).

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1. Introduction

Given a vector $x = (x_1, x_2, \dots, x_r)$ in $\mathbb{R}^r$, we write $x^\downarrow := (x_1^\downarrow, x_2^\downarrow, \dots, x_r^\downarrow)$ for the vector obtained by rearranging the components of x in the decreasing order. For two vectors $x = (x_1, x_2, \dots, x_r)$ and $y = (y_1, y_2, \dots, y_r)$ in $\mathbb{R}^r$, we say that x is *majorized* by y and write $x \prec y$ if

$$\sum_1^k x_i^\downarrow \le \sum_1^k y_i^\downarrow \quad (k = 1, 2, \dots, r-1)$$

and

$$\sum_1^r x_i^\downarrow = \sum_1^r y_i^\downarrow.$$

In matrix theory, numerous eigenvalue and trace inequalities are derived from majorization techniques. It is natural to ask if these inequalities can be extended to matrices over quaternions and octonions. The goal of this paper is to prove some eigenvalue and trace inequalities in the setting of simple Euclidean Jordan algebras via majorization techniques.

The organization of the paper is as follows. In Section 2, we cover the basic material dealing with Euclidean Jordan algebras and majorization theory. In Section 3, we prove a Thompson's triangle inequality version in simple Euclidean Jordan algebras. In Section 4, we introduce the concept of trace p-norm and study some related inequalities. In Section 5, we investigate some inequalities involving eigenvalues of sum of two objects in simple Euclidean Jordan algebras. In Section 6, we study the block type eigenvalue inequalities. In Section 7, we show some eigenvalue inequalities involving quadratic representations. In Section 8, we present more eigenvalue and trace inequalities.

2. Preliminaries

2.1. Euclidean Jordan algebras

We assume that the reader is familiar with the basic Euclidean Jordan algebra theory and recall some concepts used in this paper from Euclidean Jordan algebras. Most of these can be found in [3].

Throughout this paper, let $(V, \circ, \langle \cdot, \cdot \rangle)$ denote a Euclidean Jordan algebra: V is a finite dimensional vector space over R (the field of real numbers) with inner product $\langle x, y \rangle$ and Jordan product $x \circ y$. The symmetric cone of V is the cone of squares $K := \{x \circ x : x \in V\}$. In such an algebra, one can define the linear automorphism group $Aut(V)$ in the following way (see [3]): $\Lambda \in Aut(V)$ if $\Lambda : V \to V$ is invertible and $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all $x, y \in V$. We use the notation $x \ge 0$ $(x > 0)$ when $x \in K$ (respectively, $x \in K^o$ (=interior (K))) and $x \le 0$ $(x < 0)$ when $-x \ge 0$ $(-x > 0)$.