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Regular operator mappings and multivariate geometric means ☆

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A B S T R A C T

We introduce the notion of regular operator mappings of several variables generalising the notion of spectral function. This setting is convenient for studying maps more general than what can be obtained from the functional calculus, and it allows for Jensen type inequalities and multivariate non-commutative perspectives.

As a main application of the theory we consider geometric means of k operator variables extending the geometric mean of k commuting operators and the geometric mean of two arbitrary positive definite matrices. We propose different types of updating conditions that seems natural in many applications and prove that each of these conditions, together with a few other natural axioms, uniquely defines the geometric mean for any number of operator variables. The means defined in this way are given by explicit formulas and are computationally tractable.

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1. Introduction

The geometric mean of two positive definite operators was introduced by Pusz and Woronowicz [13], and their definition was soon put into the context of the axiomatic approach to operator means developed by Kubo and Ando [9]. Subsequently a number of authors [8,1,12,2,11,10] have suggested several ways of defining means of operators for several variables as extensions of the geometric mean of two operators.

There is no satisfactory definition of a geometric mean of several operator variables that is both computationally tractable and satisfies a number of natural conditions put forward in the influential paper by Ando, Li, and Mathias [1]. We put the emphasis on methods to extend a geometric mean of k variables to a mean of $k+1$ variables, and in the process we challenge one of the requirements to a geometric mean put forward by Ando, Li, and Mathias.

The symmetry condition of a geometric mean is mathematically very appealing, but the condition makes no sense in a number of applications. If for example positive definite matrices $A_1, A_2, \ldots, A_k$ correspond to measurements made at times $t_1 < t_2 < \cdots < t_k$ then there is no way of permuting the matrices since time only goes forward. It makes more sense to impose an updating condition

$$G_{k+1}(A_1, \ldots, A_k, 1) = G_k(A_1, \ldots, A_k)^{k/(k+1)} \tag{1}$$

when moving from a mean G_k of k variables to a mean G_{k+1} of $k+1$ variables. The condition corresponds to taking the geometric mean of k copies of $G_k(A_1, \ldots, A_k)$ and one copy of the unit matrix. A variant condition would be to impose the equality

$$G_{k+1}(A_1, \ldots, A_k, 1) = G_k\bigl(A_1^{k/(k+1)}, \ldots, A_k^{k/(k+1)}\bigr) \tag{2}$$

when updating from k to $k+1$ variables. It is an easy exercise to realise that if we set $G_1(A) = A$, then either of the conditions (1) or (2) together with homogeneity uniquely defines the geometric mean of k commuting operators.

We furthermore prove that by setting $G_1(A) = A$ and by demanding homogeneity and a few more natural conditions, then either of the updating conditions (1) or (2) leads to unique but different solutions to the problem of defining a geometric mean of k operators. The means defined in this way are given by explicit formulas, and they are computationally tractable. They possess all of the attractive properties associated with geometric operator means discussed in [1] with the notable exception of symmetry. If one emphasises either of the updating conditions (1) or (2) we are thus forced to abandon symmetry.

Efficient averaging techniques of positive definite matrices are important in many practical applications; for example in radar imaging, medical imaging, and the analysis of financial data.