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Consistent invertibility of upper triangular operator matrices ☆

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Let $\mathcal{H}$ and $\mathcal{K}$ be separable infinite dimensional Hilbert spaces, and let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be given operators. A necessary and sufficient condition is obtained for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ to be a CI operator for some (respectively, every) $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Furthermore, some related results are obtained.

1. Introduction

The study of operator matrices arises naturally from the following fact: if $\mathcal{X}$ is a Hilbert space and we decompose $\mathcal{X}$ as a direct sum of two subspaces $\mathcal{H}$ and $\mathcal{K}$, each bounded linear operator $T : \mathcal{X} \longrightarrow \mathcal{X}$ can be expressed as the operator matrix form

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$$T = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$$

with respect to the space decomposition $\mathcal{X} = \mathcal{H} \oplus \mathcal{K}$, where $A : \mathcal{H} \longrightarrow \mathcal{H}$, $B : \mathcal{K} \longrightarrow \mathcal{K}$, $C : \mathcal{K} \longrightarrow \mathcal{H}$ and $D : \mathcal{H} \longrightarrow \mathcal{K}$ are bounded linear operators, respectively. In particular, if $\mathcal{H}$ is an invariant subspace for T then $D = 0$, and so T has an upper triangular operator matrix form. One way to study operators is to see them as being composed of simpler operators. The operator matrices have been studied by numerous authors [1,2,5–7,10–12,14]. This paper is concerned with the consistent invertibility of operator matrices.

In this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable infinite dimensional Hilbert spaces. Let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$ we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we use $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* to denote the range space, the null space and the adjoint of T. For a linear subspace $\mathcal{M} \subseteq \mathcal{H}$, its closure and orthogonal complement are denoted by $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, write $n(T)$ for the nullity of T, i.e., $n(T) = \dim \mathcal{N}(T)$, and write $d(T)$ for the deficiency of T, i.e., $d(T) = \dim \mathcal{R}(T)^\perp$.

An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called a right (respectively, left) invertible operator if there exists an operator $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $TS = I_{\mathcal{K}}$ (respectively, $ST = I_{\mathcal{H}}$). If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is both left invertible and right invertible, we call it invertible. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, it is well known (see [4, pp. 347–348]) that T is right invertible if and only if T is surjective, i.e., $\mathcal{R}(T) = \mathcal{K}$, and T is left invertible if and only if $\|Tx\| \geq c\|x\|$ for every $x \in \mathcal{H}$ and some constant $c > 0$, i.e., $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) = \{0\}$.

Recall (see [9]) that an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is consistent in invertibility or, briefly, a CI operator, if, for every $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, TS and ST are invertible or non-invertible together. From [9, Remark 1.5] one can find that T is a CI operator if and only if so is T^*.

When $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given operators, we denote by M_C an operator on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

for $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The invertibility, the right and left invertibility, the semi-Fredholmness, the Weyl spectrum and Weyl's theorem of M_C were considered in [7,10,2,11], and many types of spectra for M_C were discussed in [1,5,6,12,14]. In this paper, we are mainly interested in the consistent invertibility of M_C.

2. Main results

For the proof of our main results, we need some auxiliary lemmas. We begin with:

Lemma 1. *(See [9, Theorem 1.1].) Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then T is a CI operator if and only if one of the following three mutually disjoint cases occurs:*