Linear Algebra and its Applications 455 (2014) 73–81

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The reflexivity index of a finite distributive lattice of subspaces

K.J. Harrison, J.A. Ward *

Curtin University, Perth, Australia

ARTICLE INFO

Article history:
Received 5 September 2013
Accepted 27 April 2014
Available online 20 May 2014
Submitted by P. Semrl

MSC:
47A15
47C05
06D99

Keywords:
Operator
Invariant subspace
Reflexivity
Distributive subspace lattice

ABSTRACT

We prove that a finite distributive lattice of subspaces of a vector space is reflexive and we determine its reflexivity index.

1. Introduction

Let $\mathcal{S}(X)$ denote the set of all *subspaces* of vector space X, and let $\mathcal{L}(X)$ denote the set of all *operators* on X. Here subspaces are linear manifolds and operators are linear functions that map X into itself. A subspace M of X is *invariant* under an operator T on X if $T(M) \subseteq M$, i.e. $Tx \in M$ for all $x \in M$. For any $\mathcal{L} \subseteq \mathcal{S}(X)$ and any $\mathcal{F} \subseteq \mathcal{L}(X)$ we define

$$\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{L}(X) : T(M) \subseteq M \text{ for all } M \in \mathcal{L}\}, \quad \text{and}$$

* Corresponding author.

http://dx.doi.org/10.1016/j.laa.2014.04.026

$$\operatorname{Lat}\mathcal{F} = \{M \in \mathcal{S}(X) : T(M) \subseteq M \text{ for all } T \in \mathcal{F}\}.$$

That is, $\operatorname{Alg}\mathcal{L}$ is the set of operators on X that leave each subspace in $\mathcal{L}$ invariant, and $\operatorname{Lat}\mathcal{F}$ is the set of all subspaces of X that are invariant under each operator in $\mathcal{F}$.

We say a subset $\mathcal{L}$ of $\mathcal{S}(X)$ is *reflexive* if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$. Since $\operatorname{Lat}\mathcal{F} = \operatorname{Lat}\operatorname{Alg}\operatorname{Lat}\mathcal{F}$ for any $\mathcal{F} \subseteq \mathcal{L}(X)$, it follows that $\mathcal{L}$ is reflexive if and only if $\mathcal{L} = \operatorname{Lat}\mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{L}(X)$. Similarly, a subset $\mathcal{F}$ of $\mathcal{L}(X)$ is *reflexive* if $\mathcal{F} = \operatorname{Alg}\operatorname{Lat}\mathcal{F}$. Since $\operatorname{Alg}\mathcal{L} = \operatorname{Alg}\operatorname{Lat}\operatorname{Alg}\mathcal{L}$ for any $\mathcal{L} \subseteq \mathcal{S}(X)$, $\mathcal{F}$ is reflexive if and only if $\mathcal{F} = \operatorname{Alg}\mathcal{L}$ for some $\mathcal{L} \subseteq \mathcal{S}(X)$.

The set $\mathcal{S}(X)$ of all subspaces is a complete lattice under the operations $\bigwedge\{M_\alpha : \alpha \in \Omega\} = \bigcap\{M_\alpha : \alpha \in \Omega\}$ and $\bigvee\{M_\alpha : \alpha \in \Omega\} = \operatorname{span}\{M_\alpha : \alpha \in \Omega\}$, where $\{M_\alpha : \alpha \in \Omega\}$ is any family of subspaces of X, and $\operatorname{span}\{M_\alpha : \alpha \in \Omega\}$ denotes the set of all finite linear combinations of vectors in the subspaces M_α. For any family $\mathcal{F}$ of operators on X, $\operatorname{Lat}\mathcal{F}$ is closed under arbitrary intersections and linear spans and contains the trivial subspaces $\{0\}$ and X. So a reflexive family of subspaces of X is necessarily a complete sublattice of $\mathcal{S}(X)$ containing the trivial subspaces. In the remainder of this paper it will be implicitly assumed that any lattice of subspaces is complete and contains the trivial subspaces.

It is well known that $\mathcal{S}(X)$, and each of its sublattices, is modular. However $\mathcal{S}(X)$ is not distributive if $\dim X > 1$, because if M_1, M_2 and M_3 are distinct lines (i.e. 1-dimensional subspaces of X) contained in a plane (i.e. a 2-dimensional subspace of X), then

$$M_1 = M_1 \wedge (M_2 \vee M_3) \neq (M_1 \wedge M_2) \vee (M_1 \wedge M_3) = \{0\}$$

Nevertheless, there are distributive sublattices of $S(X)$. We shall show that each finite disributive sublattice is reflexive.

The notion of reflexivity was introduced by Halmos [2,3] in a different context. In those papers the vector space X is a Hilbert space, the subspaces are closed and the operators are bounded. Many of the early results concerning reflexivity of lattices of closed subspaces involve distributivity. For example, it is shown in [3] that every complete atomic Boolean algebra of subspaces is reflexive. It had already been shown by Ringrose [7] that every complete nest of subspaces is reflexive, and by Johnson [5] that every distributive lattice of subspaces of a finite-dimensional vector space is reflexive. Johnson's result was extended by Harrison [4] who showed that every finite distributive lattice of subspaces of a Hilbert space of any dimension is reflexive. These results were generalised by Longstaff [6] who showed that completely distributive subspace lattices are reflexive. Complete distributivity is a strong form of distributivity concerning arbitrarily many meets and joins. See [6] for the precise definition. Arveson [1] has shown that separably acting commutative subspace lattices are reflexive. A commutative subspace lattice is one in which the orthogonal projections corresponding to the subspaces commute and form a set that is closed in the strong operator topology. It is separably acting if the underlying Hilbert space is separable. Commutative subspace lattices are distributive, but not necessarily completely distributive.