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Universal state transfer on graphs

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A B S T R A C T

A continuous-time quantum walk on a graph G is given by the unitary matrix $U(t) = \exp(-itA)$, where A is the adjacency matrix of G. We say G has pretty good state transfer between vertices a and b if for any $\epsilon > 0$, there is a time t, where the (a, b)-entry of $U(t)$ satisfies $|U(t)_{a,b}| \geq 1 - \epsilon$. This notion was introduced by Godsil (2011). The state transfer is perfect if the above holds for $\epsilon = 0$. In this work, we study a natural extension of this notion called universal state transfer wherein state transfer exists between every pair of vertices of the graph. We prove the following results about graphs with this stronger property: (i) Graphs with universal state transfer have distinct eigenvalues and flat eigenbasis (each eigenvector has entries which are equal in magnitude). (ii) The switching automorphism group of a graph with universal state transfer is abelian and its order divides the size of the graph. Moreover, if the state transfer is perfect, the switching automorphism group is cyclic. (iii) There is a family of complex oriented prime-length cycles which has universal pretty good state transfer. This provides a concrete example of a family of graphs with this universal property. (iv) There exists a class of graphs with real symmetric adjacency matrices which has universal pretty good state transfer. In contrast, Kay (2011)

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proved that no graph with real-valued adjacency matrix can have universal perfect state transfer. Finally, we provide a spectral characterization of universal perfect state transfer for graphs switching equivalent to circulants.

1. Introduction

The study of continuous-time quantum walk on graphs is important for several reasons. Originally, it was studied for developing new quantum algorithmic techniques. This led to the seminal results of Childs et al. [1] and of Farhi et al. [2]. The algorithms developed in [1,2] are notable in that they provably beat the corresponding classical resource bounds. The main goal here is to develop applications for quantum walk algorithms which have the same impact as Shor's quantum algorithm for factoring integers.

Around the same time, Bose [3] studied the problem of quantum information transmission in quantum spin chains. As stated by Bose, this problem may be viewed as a continuous-time quantum walk on a path where perfect state transfer occurs between the two antipodal endpoints. Subsequently, Christandl et al. [4,5] proved that perfect state transfer between antipodal points on a path with n vertices is only possible whenever $n = 2$ or 3. Despite this negative result, the problem of perfect state transfer on other finite graphs became an interesting question in both quantum information and algebraic graph theory. A recent survey in this area is given by Godsil [6].

More recently, Childs [7] showed that quantum walk is a key ingredient in simulating universal quantum computation. Later, Underwood and Feder [8] showed how to use perfect state transfer as an alternative to the graph scattering methods used by Childs [7]. This underscores the importance of perfect state transfer in quantum walks.

Several works have raised the difficulties in requiring the state transfer be perfect (for example, Anderson localization [9]). This led to the natural notion of *pretty good state transfer* which was introduced by Godsil [6]. In a recent breakthrough, Godsil, Kirkland, Severini and Smith [10] proved that some families of paths (whose lengths satisfy specific number-theoretic conditions) have pretty good state transfer. This is in contrast to the negative result of Christandl et al. [4,5] and it confirmed some of the numerical observations made by Bose [3]. A further result on pretty good state transfer was given by Fan and Godsil [11] on the so-called double-star graphs.

A fundamental observation by Kay [12] is that perfect state transfer between two non-disjoint pairs of vertices is impossible for graphs with real symmetric adjacency matrices. In this work, we study and exhibit graphs with complex Hermitian adjacency matrices which have *universal* state transfer. By universal, we mean that state transfer exists between *every* pair of vertices. Graphs with complex Hermitian adjacency matrices are also known as complex gain graphs – which are generalizations of signed graphs (see Zaslavsky [13]). For brevity, we will refer to these graphs as *Hermitian* graphs. As the