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A flag representation for finite collections of subspaces of mixed dimensions

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ABSTRACT

Given a finite set of subspaces of $\mathbb{R}^n$, perhaps of differing dimensions, we describe a flag of vector spaces (i.e. a nested sequence of vector spaces) that best represents the collection based on a natural optimization criterion and we present an algorithm for its computation. The utility of this flag representation lies in its ability to represent a collection of subspaces of differing dimensions. When the set of subspaces all have the same dimension d, the flag mean is related to several commonly used subspace representations. For instance, the d-dimensional subspace in the flag corresponds to the extrinsic manifold mean. When the set of subspaces is both well clustered and equidimensional of dimension d, then the d-dimensional component of the flag provides an approximation to the Karcher mean. An intermediate matrix used to construct the flag can also be used to recover the canonical components at the heart of Multiset Canonical Correlation Analysis. Two examples utilizing the Carnegie Mellon University Pose, Illumination, and Expression Database (CMU-PIE) serve as visual illustrations of the algorithm.

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1. Introduction

The Grassmann manifold has found use as a setting in which to classify and make comparisons between large data sets. It is particularly effective when aspects of the data can be captured with linear subspaces. A sampling of settings where Grassmann techniques have been applied includes activity modeling and recognition, shape analysis, action classification, face recognition, person detection, subspace tracking, and general manifold clustering [14,10,12,4–6,19,20,18]. Given a cluster of points on a Grassmann manifold, algorithms have been developed to find a point on the manifold which represents the cluster [5,1,7,17]. These cluster representatives play the role of an *average subspace* and can be used to reduce the cost of classification algorithms or to aid in clustering tasks.

In a more general setting, consider data consisting of subspaces of $\mathbb{R}^n$ of differing dimensions, i.e. a data cloud living on a disjoint union of Grassmann manifolds. This paper proposes a *flag mean* representation for such a collection. The flag mean is a nested sequence of vector spaces that best fits the data according to an optimization criterion based on the projection Frobenius norm. The subspaces in the flag can be treated independently as points on Grassmann manifolds, or collectively as a single point on a *flag manifold*.

The layout of the paper is as follows. Section 2 provides background, definitions and motivation for the construction. Section 3 presents the optimization problem, whose solution is the flag mean, and provides an analytical solution by the method of Lagrange multipliers. The result is an ordered set of unit length vectors most central to the collection of subspaces being averaged. Section 4 exploits the singular value decomposition as an efficient computational tool for determining this ordered set of unit length vectors and connects them to multiset canonical correlation analysis. In Section 5 the central vectors are used to construct the flag mean. The construction is then illustrated with two numerical experiments using data drawn from the Carnegie Mellon University Pose, Illumination, and Expression Database (CMU-PIE). Section 6 explores a special case of the flag mean and relates it to alternative subspace means found in the literature. Section 7 discusses conclusions and future work.

2. Background

Many image and video based computer vision systems represent data as a set of linear subspaces of a fixed dimension [17,19,6,4,1]. This structure allows the data to be treated as a collection of points on a single Grassmann manifold. The Grassmann manifold $\mathrm{Gr}(V, p)$ is a manifold whose points parametrize the subspaces of dimension p inside the