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Signature matrix algebras and bipartite graphs

Valeria Aguirre Holguín [a], Piotr J. Wojciechowski [b,*]

[a] *Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001, United States*
[b] *Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, United States*

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ABSTRACT

A special class of matrix algebras, the rc-signature algebras, naturally emerged as a result of the study of a Multiplicative Decomposition Property of matrices (a multiplicative analogue of the Riesz Decomposition Property in ordered vector spaces). This note is devoted to the study of a tractable subclass of these algebras. It is proven that a necessary and sufficient condition for two such algebras to be isomorphic is the simultaneous permutation-similarity between the members of the algebras. There is a one-to-one correspondence between the signature algebras and the bipartite graphs that respects the isomorphism between the algebras and the strict isomorphism between the bipartite graphs.

1. Preliminaries

In [3] were studied subalgebras of the matrix algebra $M_n(\mathbb{R})$ that satisfy a *multiplicative decomposition* property, which states that

$$\text{if } A, B \geqslant \mathbf{0} \text{ and } \mathbf{0} \leqslant C \leqslant AB$$
$$\text{then there exist } A', B' \geqslant \mathbf{0} \text{ such that } A' \leqslant A,\ B' \leqslant B \text{ and } C = A'B'$$

* Corresponding author.
E-mail addresses: vah@nmsu.edu (V. Aguirre Holguín), piotrw@utep.edu (P.J. Wojciechowski).

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where all the matrices mentioned are from the considered subalgebra, **0** is the zero matrix and the order is the usual entry-wise lattice order (see also [4]).

It was proven [3, Corollary 3.5] that every such algebra embeds in a so-called signature algebra (defined below). The current paper is devoted to the study of the signature algebras and their combinatorial nature. After introducing some basic definitions and facts, we proceed to Section 2 which provides necessary and sufficient conditions for two such algebras to be isomorphic, followed by some corollaries. The combinatorial character of these conditions emerges together with a certain permutation accounting for the isomorphism (Theorem 2.1 and Corollary 2.2). This observation leads to the strict connection with the bipartite graphs we can assign to the signature algebras. We explore this issue in Section 3. All technical terms and basic facts related to matrices used here can be found in most linear algebra books.

Definition 1.1. We say that an $n \times n$ matrix has a *signature* $\sigma = (s_i)$ if (s_i) is an n-element sequence with $s_i \in \{r, c\}$, where $s_i = r$ means that for all $j \neq i$, $a_{ij} = 0$, similarly $s_i = c$ means that for all $j \neq i$, $a_{ji} = 0$. Consequently, a matrix algebra $\mathcal{A}$ has a signature σ if it is a subalgebra (always with the identity I) of $M_n(\mathbb{R})$ such that every matrix from $\mathcal{A}$ has the signature σ.

The following rules can be easily computed within a signature algebra:

(i) if $C = AB$, then

$$c_{ij} = \begin{cases} a_{ii}b_{ij} + a_{ij}b_{jj} & \text{if } i \neq j, \\ a_{ii}b_{jj} & \text{if } i = j, \end{cases}$$

(ii) if A is a nonsingular signature matrix then its inverse B is given by:

$$b_{ij} = \begin{cases} b_{ii} = \frac{1}{a_{ii}} & \text{for } i = j, \\ b_{ij} = -\frac{a_{ij}}{a_{ii}a_{jj}} & \text{for } i \neq j, \end{cases}$$

(iii) the determinant of a signature matrix is the product of its diagonal entries.

It follows from (i) that the off-diagonal part

$$\mathcal{N}(\mathcal{A}) = \{A \in \mathcal{A}\colon\ a_{ii} = 0,\ i = 1, \ldots, n\}$$

is an algebra with zero multiplication.

Several other properties and examples of the signature algebras were discussed in [1] and [3]. Here is one.