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Evaluation modules for the q-tetrahedron algebra

Tatsuro Ito [a], Hjalmar Rosengren [b], Paul Terwilliger [c,*]

[a] Division of Mathematical and Physical Sciences, Graduate School of Natural Science and Technology, Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Japan

[b] Department of Mathematical Sciences, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

[c] Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706-1388, USA

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ABSTRACT

Let $\mathbb{F}$ denote an algebraically closed field, and fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. We consider the q-tetrahedron algebra $\boxtimes_q$ over $\mathbb{F}$. It is known that each finite-dimensional irreducible $\boxtimes_q$-module of type 1 is a tensor product of evaluation modules. This paper contains a comprehensive description of the evaluation modules for $\boxtimes_q$. This description includes the following topics. Given an evaluation module V for $\boxtimes_q$, we display 24 bases for V that we find attractive. For each basis we give the matrices that represent the $\boxtimes_q$-generators. We give the transition matrices between certain pairs of bases among the 24. It is known that the cyclic group $\mathbb{Z}_4$ acts on $\boxtimes_q$ as a group of automorphisms. We describe what happens when V is twisted via an element of $\mathbb{Z}_4$. We discuss how evaluation modules for $\boxtimes_q$ are related to Leonard pairs of q-Racah type.

* Corresponding author.
E-mail addresses: ito@se.kanazawa-u.ac.jp (T. Ito), hjalmar@chalmers.se (H. Rosengren), terwilli@math.wisc.edu (P. Terwilliger).

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1. Introduction

The q-tetrahedron algebra $\boxtimes_q$ was introduced in [10]. This algebra is associative, noncommutative, and infinite-dimensional. It is defined by generators and relations. There are eight generators, and it is natural to identify each of these with an orientation on an edge in a tetrahedron. From this point of view the generating set looks as follows. In the tetrahedron, a pair of opposite edges are each oriented in both directions. The four remaining edges are each oriented in one direction, to create a directed 4-cycle. Thus the cyclic group $\mathbb{Z}_4$ acts transitively on the vertex set of the tetrahedron, in a manner that preserves the set of edge-orientations. The relations in $\boxtimes_q$ are described as follows. For each doubly oriented edge of the tetrahedron, the product of the two edge-orientations is 1. For each pair of edge-orientations that create a directed 2-path involving three distinct vertices, these edge-orientations satisfy a q-Weyl relation. For each pair of edges in the tetrahedron that are opposite and singly oriented, the two associated edge-orientations satisfy the cubic q-Serre relations. By construction, the $\mathbb{Z}_4$ action on the tetrahedron induces a $\mathbb{Z}_4$ action on $\boxtimes_q$ as a group of automorphisms.

We will be discussing the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$, the loop algebra $U_q(L(\mathfrak{sl}_2))$ [10, Section 8], and an algebra $\mathcal{A}_q$ called the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$ [10, Definition 9.1]. These algebras are related to $\boxtimes_q$ in the following way. Each face of the tetrahedron is surrounded by three edges, of which two are singly oriented and one is doubly oriented. The resulting four edge-orientations generate a subalgebra of $\boxtimes_q$ that is isomorphic to $U_q(\mathfrak{sl}_2)$ [10, Proposition 7.4], [21, Proposition 4.3]. Upon removing one doubly oriented edge from the tetrahedron, the remaining six edge-orientations generate a subalgebra of $\boxtimes_q$ that is isomorphic to $U_q(L(\mathfrak{sl}_2))$ [21, Proposition 4.3]. For each pair of edges in the tetrahedron that are opposite and singly oriented, the two associated edge-orientations generate a subalgebra of $\boxtimes_q$ that is isomorphic to $\mathcal{A}_q$ [21, Proposition 4.1].

The above containments reveal a close relationship between the representation theories of $\boxtimes_q$, $U_q(L(\mathfrak{sl}_2))$, and $\mathcal{A}_q$. Before discussing the details, we comment on $U_q(L(\mathfrak{sl}_2))$. In [1], Chari and Pressley classify up to isomorphism the finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules. This classification involves a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules of type 1; (ii) the polynomials in one variable that have constant coefficient 1. The polynomial is called the Drinfel'd polynomial.

The representation theories for $\boxtimes_q$ and $U_q(L(\mathfrak{sl}_2))$ are related as follows. Let V denote a $\boxtimes_q$-module. Earlier we mentioned a subalgebra of $\boxtimes_q$ that is isomorphic to $U_q(L(\mathfrak{sl}_2))$. Upon restricting the $\boxtimes_q$ action on V to this subalgebra, V becomes a $U_q(L(\mathfrak{sl}_2))$-module. The restriction procedure yields a map from the set of $\boxtimes_q$-modules to the set of $U_q(L(\mathfrak{sl}_2))$-modules. By [7, Remark 1.8] and [10, Remark 10.5], this map induces a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1; (ii) the isomorphism classes of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-modules of type 1 whose associated Drinfel'd polynomial does not vanish at 1. (We follow the normalization conventions from [21].) In [21], Miki extends