Linear Algebra and its Applications 451 (2014) 182–196

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Walks and cycles on a digraph with application to population dynamics

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ARTICLE INFO

Article history:
Received 31 December 2013
Accepted 3 March 2014
Available online 2 April 2014
Submitted by R. Brualdi

MSC:
05C20
92D30

Keywords:
Spectral radius
Target reproduction number
Closed walk
Digraph
Infectious disease control

ABSTRACT

Several relations involving closed walks and closed cycles in a weighted digraph are established. These relations are used to derive new expressions for target reproduction numbers for controlling the spectral radius of nonnegative matrices. The results are illustrated by an application to infectious disease control.

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http://dx.doi.org/10.1016/j.laa.2014.03.003

1. Introduction

Let $W = [w_{ij}]$ be a nonnegative $n \times n$ matrix. Its spectral radius, denoted by $\rho(W)$, gives the asymptotic growth rate of any matrix norm of W^k (see, for example, [5] and [7, Section 5.6]). This fact is of wide application in population dynamics. For example, in a Leslie matrix population model [3, Chapter 2], the spectral radius of the nonnegative projection matrix W determines whether the population grows or goes to extinction, depending on whether $\rho(W) > 1$ or $0 \leqslant \rho(W) < 1$. In an infectious disease model [1,4] with a nonnegative next-generation matrix W, the spectral radius $\rho(W)$, called the basic reproduction number in this content, often determines whether the disease persists or dies out, depending on whether $\rho(W) > 1$ or $\rho(W) < 1$. A natural question is how to reduce or enlarge one or more entries of W so that the controlled matrix W_c (that is obtained from W by reducing or enlarging those entries of W) has spectral radius 1. It turns out that the target reproduction number defined in [11] can be used to measure the magnitude of this reduction or enlargement.

Suppose that only one entry of W is targeted, say the entry w_{ij} for some i, j with $1 \leqslant i, j \leqslant n$. Let A denote the matrix obtained from W by replacing the entry w_{ij} by zero, i.e.,

$$A = W - P_i W P_j, \tag{1.1}$$

where P_i is the $n \times n$ projection matrix with the (i, i) entry equal to 1 and all other entries 0. If $\rho(A) < 1$, then the target reproduction number T_{ij} [11] is defined as

$$T_{ij} = w_{ij}(I - A)^{-1}_{ji}, \tag{1.2}$$

where I is the $n \times n$ identity matrix. To exclude trivial situations, we always assume that $\rho(W) > 0$ and that $T_{ij} > 0$ whenever it is well defined. The following special case of Theorem 4.4 in [11] gives a relation between $\rho(W)$ and T_{ij}.

Proposition 1.1. *Let W be an $n \times n$ nonnegative irreducible matrix such that $\rho(A) < 1$, where A is as defined in (1.1). Then either $1 < \rho(W) < T_{ij}$, $\rho(W) = T_{ij} = 1$, or $T_{ij} < \rho(W) < 1$.*

The next result, a special case of Theorem 2.2 in [11], shows how T_{ij} can be used to modify w_{ij} to find the controlled matrix W_c whose spectral radius equals 1.

Proposition 1.2. *Let W be an $n \times n$ nonnegative irreducible matrix such that $\rho(A) < 1$, where A is as defined in (1.1). If W_c is the controlled matrix obtained from W by replacing w_{ij} by w_{ij}/T_{ij}, then $\rho(W_c) = 1$.*

Example 1. Suppose $W = \left[\begin{smallmatrix} 0.5 & 0.7 \\ 0.7 & 0.5 \end{smallmatrix}\right]$; then $\det(\lambda I - W) = (\lambda + 0.2)(\lambda - 1.2)$, and thus $\rho(W) = 1.2$. In this case, it follows from (1.2) that $T_{11} = 0.5\left[\begin{smallmatrix} 1 & -0.7 \\ -0.7 & 1-0.5 \end{smallmatrix}\right]^{-1}_{11} =$