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# New bounds for roots of polynomials based on Fiedler companion matrices ☆

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ABSTRACT

Several matrix norms of the classical Frobenius companion matrices of a monic polynomial $p(z)$ have been used in the literature to obtain simple lower and upper bounds on the absolute values of the roots $\lambda$ of $p(z)$. Recently, M. Fiedler (2003) [9] has introduced a new family of companion matrices of $p(z)$ that has received considerable attention and it is natural to investigate if matrix norms of Fiedler companion matrices may be used to obtain new and sharper lower and upper bounds on $|\lambda|$. The development of such bounds requires first to know simple expressions for some relevant matrix norms of Fiedler matrices and we obtain them in the case of the 1- and $\infty$-matrix norms. With these expressions at hand, we will show that norms of Fiedler matrices produce many new bounds, but that none of them improves significatively the classical bounds obtained from the Frobenius companion matrices. However, we will prove that if the norms of the inverses of Fiedler matrices are used, then another family of new bounds on $|\lambda|$ is obtained and some of the bounds in this family improve significatively the bounds coming from the Frobenius companion matrices for certain polynomials.

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## 1. Introduction

This paper is devoted to find bounds for the absolute values of the roots of $p(z)$, a monic polynomial of degree $n \geqslant 2$ with complex coefficients written as

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad a_i \in \mathbb{C},\ i = 0, 1, \ldots, n-1. \tag{1}$$

To locate approximately the roots of $p(z)$ through simple operations with its coefficients is a classical problem that has produced a considerable amount of literature (see the comprehensive surveys [15,17] and the references therein). Simple location rules are used for theoretical purposes, as establishing sufficient conditions guaranteeing that $p(z)$ is stable or that all its roots are inside the unit circle, and they are also used in iterative algorithms for computing the roots of $p(z)$ to find initial guesses of the roots for starting the iteration [2,3]. Recently, polynomial eigenvalue problems have received much attention and simple criteria for locating approximately the eigenvalues of matrix polynomials have been developed [4,12], but, to keep the paper concise, matrix polynomials are not covered in this work.

Let us denote by $\lambda$ any root of $p(z)$. In this paper, we are interested in finding non-negative numbers $L(p)$ and $U(p)$ depending on the coefficients of $p(z)$, such that

$$L(p) \leqslant |\lambda| \leqslant U(p), \tag{2}$$

by using norms of the Fiedler matrices associated with $p(z)$. The Fiedler matrices [9] of $p(z)$ are a family of $2^{n-1}$ different matrices whose eigenvalues are precisely the roots of $p(z)$. The family of Fiedler matrices has received considerable attention in the last years, and it includes the well-known *first* and *second (Frobenius) companion forms* of $p(z)$, that is, the matrices

$$C_1(p) = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_2(p) = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3}$$

which have been widely used to obtain classic bounds of type (2) [13, pp. 365–368], as well as other types of location results for roots of polynomials [16]. However, to the best of our knowledge, other Fiedler matrices have not yet been used for these purposes and this is the goal of this paper.