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Interval eigenproblem in max–min algebra

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ABSTRACT

Interval eigenvectors of interval matrices in max–min algebra are investigated. The characterization of interval eigenvectors which has been presented in [7] for increasing eigenvectors, is extended here to general interval eigenvectors. Classification types of general interval eigenvectors are studied and characterization of all possible six types is presented.

1. Introduction

The input data in real problems are usually not exact and can be rather characterized by interval values. Considering matrices and vectors with interval coefficients is therefore of great practical importance, see [2,3,5,8,9]. For systems described by interval coefficients the investigation of steady states leads to computing interval eigenvectors. The eigenspace structure of a given interval matrix A in max–min algebra is studied in the paper.

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By max–min algebra we understand a triple $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{B}$. The notation $\mathcal{B}(m,n)$ ($\mathcal{B}(n)$) denotes the set of all matrices (vectors) of given dimension over $\mathcal{B}$. Operations $\oplus, \otimes$ are extended to addition and multiplication of matrices and vectors in the standard way. We assume that $\mathcal{B}$ contains the minimal element O and the maximal element I.

The linear order on $\mathcal{B}$ extends componentwise to partial ordering on $\mathcal{B}(m,n)$ and $\mathcal{B}(n)$ and the notation $\vee$ ($\wedge$) is used for the binary join (meet) in these sets. These operators may also be quantified over a set S, as $\bigvee_{i \in S} x_i$ (resp. $\bigwedge_{i \in S} x_i$) whenever the upper (resp. lower) bound exists. The Boolean notation $\cup$ and $\cap$ respectively will be used for union and intersection of sets.

The eigenproblem for a given matrix $A \in \mathcal{B}(n,n)$ in max–min algebra consists in finding a value $\lambda \in \mathcal{B}$ (eigenvalue) and a vector $x \in \mathcal{B}(n)$ (eigenvector) such that the equation $A \otimes x = \lambda \otimes x$ holds true. It is well-known that the above problem in max–min algebra can be reduced to solving the equation $A \otimes x = x$. The eigenproblem in max–min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace (the set of all eigenvectors), and algorithms for computing the greatest eigenvector of a given matrix were suggested, see e.g. [1,4].

The complete structure of the eigenspace has been described in [6], where the eigenspace of a given matrix is presented as a union of intervals of permuted monotone eigenvectors. Explicit formulas are shown for the lower and upper bounds of the intervals of monotone eigenvectors for any given monotonicity type. By permutation of indices, the formulas are then used for description of the whole eigenspace of a given matrix. The monotonicity approach from [6] has been applied to the interval eigenproblem in [7], where a classification consisting of six different types of interval eigenvectors is presented, and detailed characterization of these types is given for increasing interval eigenvectors (which are called 'strictly increasing' in both papers [6,7]).

The aim of this paper is to extend the results from [7] to all types of non-decreasing interval eigenvectors (which are simply called 'increasing' in [6] and [7]). Furthermore, permutations of indices are used in the paper for finding characterizations of all six classification types without any restriction. By this, the interval eigenproblem is completely solved.

The rest of the paper is organized as follows. Section 2 contains a simple application of max–min eigenvectors. In Section 3 the basic results from [6] and the necessary notions are described, which will be used later in Section 5. In Section 4 we define interval matrices and six basic types of interval eigenvectors, according to [7]. Section 5 is the main section: for a given interval partition D, all six types of D-increasing interval eigenvectors are described. The results are generalized for arbitrary interval eigenvectors in Section 6. Finally, Section 7 shows all relations between the considered six general types of interval eigenvectors.

2. Max–min eigenvectors in applications

Eigenvectors of matrices in max–min algebra are useful in applications such as automata theory, design of switching circuits, logic of binary relations, medical diagnosis, Markov chains, social choice, models of organizations, information systems, political systems and clustering.

As an example of a business application we consider evaluation of projects adopted by a company. Position of a project is characterized by several components, such as its importance for the future of the company, the assigned investments, the intensity of the project, or its impact on market. The level of each component i is described by some value x_i, which is influenced by the levels of all components x_j. The influence is expressed by a constant factor a_{ij}, and the position vector x at time $t+1$ is given by equalities $x_i(t+1) = \max_j(\min(a_{ij}, x_j(t))) = \bigoplus_j (a_{ij} \otimes x_j(t))$ for every i, or shortly $x(t+1) = A \otimes x(t)$, in matrix notation. The steady position of the project is then described by the equation $x(t+1) = x(t)$, i.e. $A \otimes x(t) = x(t)$. In other words, $x(t)$ is then an eigenvector of matrix A, which means that the steady states exactly correspond to eigenvectors. In reality, interval eigenvectors and interval matrices should be considered.

Further examples are related to business applications describing other activities, e.g. planning different forms of advertisement. Similar applications can also be found in social science, biology, medicine, computer nets, and many other areas.