Linear Algebra and its Applications 440 (2014) 50–60

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The absolute bound for coherent configurations

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ARTICLE INFO

Article history:
Received 31 October 2012
Accepted 27 October 2013
Available online 19 November 2013
Submitted by R. Brualdi

MSC:
05E30
05z5

Keywords:
Association scheme
Coherent configuration
Absolute bound

ABSTRACT

In this paper we generalize the absolute bound for association schemes to coherent configurations. We examine this bound in the context of quasi-symmetric and strongly regular designs. In particular, we use it to derive a new feasibility condition for strongly regular designs and give examples of parameter sets ruled out by this condition but which pass all other feasibility conditions known to us.

1. Introduction and definitions

Coherent configurations are, simply put, association schemes linked by relations which satisfy certain regularity conditions. Commutative association schemes have received the greatest amount of attention, and many objects have been modeled as association schemes in design theory, finite geometry and coding theory. However, some objects are better modeled as coherent configurations. For example, generalized quadrangles (see [15]) give rise to strongly regular graphs, but with the exception of those of order (q, q^2) they cannot be characterized by strongly regular graphs. They can, however, be characterized as certain coherent configurations. There are also instances when an association scheme can be fissioned into a coherent configuration to obtain more information. This was successfully used by Schrijver [16] to give new bounds on error correcting codes.

This paper is part of ongoing work to generalize the theory of commutative association schemes to all coherent configurations. One way to do this is to apply the standard theory to each commutative association scheme contained in the configuration. However, this rather trivial generalization

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.10.045

does not exploit the fact that inside the coherent configuration, these schemes must also have a nice relationship with one another, and it does not apply to non-commutative fibers.

This suggests that stronger results should be possible. The first such result came when the Krein conditions were generalized by Hobart in [10] to give a stronger necessary condition. Later, Hobart generalized Delsarte's bounds on subsets [11].

In this paper, we consider the absolute bound. In an association scheme, the absolute bound is an inequality between the ranks of entrywise products of idempotents and sums of eigenvalue multiplicities. This condition has been used to rule out distance regular graphs (in particular certain strongly regular graphs, see [2]).

We present a generalization of this bound to coherent configurations. We also give examples of feasible parameters for configurations ruled out by this condition.

First, we begin with some definitions.

A *coherent configuration* (c.c.) is a pair $(X, \mathcal{R})$ such that X is a finite set and $\mathcal{R} = \{R_1, \ldots, R_d\}$ with $R_i \subset X \times X$, satisfying the following:

1. $\mathcal{R}$ is a partition of $X \times X$.
2. If $R_i \cap I \neq \emptyset$ then $R_i \subset I$, where $I = \{(x, x)\colon\ x \in X\}$.
3. For each $R_i \in \mathcal{R}$, $R_i^T \in \mathcal{R}$, where $R_i^T = \{(y, x) \in X \times X\colon\ (x, y) \in R_i\}$.
4. For $R_i, R_j, R_k \in \mathcal{R}$ and $x, y \in X$ with $(x, y) \in R_k$, the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k, independent of the choice of x and y.

Each relation R_i has a corresponding $|X| \times |X|$ matrix A_i, defined by $(A_i)_{xy} = 1$ or 0 depending on whether $(x, y) \in R_i$ or not. The matrices $\{A_i\}$ generate a semisimple matrix algebra $\mathfrak{A}$ over the complex numbers, which is also closed under entrywise multiplication.

The partition of the identity relation yields a partition of X into sets X_α, $\alpha \in \Omega$, called the *fibers* of the c.c. For fibers X_α and X_β, define the subspace $\mathfrak{A}_{\alpha\beta}$ to be the set of all $A \in \mathfrak{A}$ such that for all $x, y \in X$ we have $A_{xy} \neq 0$ implies that $x \in X_\alpha$ and $y \in X_\beta$. Let $\mathfrak{A}_\alpha = \mathfrak{A}_{\alpha\alpha}$; note that $\mathfrak{A}_\alpha$ is a subalgebra of $\mathfrak{A}$.

If a coherent configuration has only one fiber, it is called an *association scheme*. Note that there is an induced association scheme on each fiber X_α whose Bose–Mesner algebra is isomorphic to $\mathfrak{A}_\alpha$; we will also refer to these schemes as fibers of the c.c. Coherent configurations are, in a sense, association schemes linked by additional relations. If $\mathfrak{A}$ is commutative, the c.c. must be an association scheme, which we call a *commutative* association scheme. For more information on commutative association schemes, see [3].

For the corresponding theory of coherent configurations, see [7] and [8]; we will in general use the notation of [8]. Let $\Delta = \{\Delta_s\colon\ s \in S\}$ be the set of absolutely irreducible representations of $\mathfrak{A}$ over the complex numbers. We choose the representations so that $\Delta_s(A^*) = (\Delta_s(A))^*$, where $*$ denotes the conjugate transpose. Denote the degree of the representation Δ_s by e_s, and the multiplicity by h_s.

Since $\mathfrak{A}$ is semisimple, it decomposes into a direct sum of algebras $\bigoplus \mathfrak{C}_s$ where each algebra $\mathfrak{C}_s$ is isomorphic to $M_{e_s}(\mathbb{C})$. We have the following result about ranks of matrices in $\mathfrak{A}$.

Lemma 1. *For all $A \in \mathfrak{A}$,*

$$\operatorname{rk}(A) = \sum_{s \in S} h_s \operatorname{rk}\bigl(\Delta_s(A)\bigr).$$

Proof. If we block diagonalize the matrix algebra, the resulting matrix A' has exactly h_s blocks which are equivalent to $\Delta_s(A)$. The formula for the rank of A then follows. □

We can find a basis $\mathcal{E}_{ij}^s$ for each algebra $\mathfrak{C}_s$, satisfying

$$\mathcal{E}_{ij}^s \mathcal{E}_{kl}^t = \delta_{st}\delta_{jk}\mathcal{E}_{il}^s, \qquad \bigl(\mathcal{E}_{ji}^s\bigr)^* = \mathcal{E}_{ij}^s, \qquad \Delta_s\bigl(\mathcal{E}_{ij}^t\bigr) = \delta_{st} E_{ij}^s \tag{1}$$

where E_{ij}^s is the $e_s \times e_s$ matrix with a 1 in the i, j entry and 0 everywhere else.