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Characterization of tropical hemispaces by (P,R)-decompositions

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ABSTRACT

We consider tropical hemispaces, defined as tropically convex sets whose complements are also tropically convex, and tropical semispaces, defined as maximal tropically convex sets not containing a given point. We introduce the concept of (P,R)-decomposition. This yields (to our knowledge) a new kind of representation of tropically convex sets extending the classical idea of representing convex sets by means of extreme points and rays. We characterize tropical hemispaces as tropically convex sets that admit a (P,R)-decomposition of certain kind. In this characterization, with each tropical hemispace we associate a matrix with coefficients in the completed tropical semifield, satisfying an extended rank-one condition. Our proof techniques are based on homogenization (lifting a convex set to a cone), and the relation between tropical hemispaces and semispaces.

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1. Introduction

Max-plus algebra is the algebraic structure obtained when considering the *max-plus semifield* $\mathbb{R}_{\max,+}$. This semifield is defined as the set $\mathbb{R} \cup \{-\infty\}$ endowed with $\alpha \oplus \beta := \max(\alpha, \beta)$ as addition and the usual real numbers addition $\alpha \otimes \beta := \alpha + \beta$ as multiplication. Thus, in the max-plus semifield, the neutral elements for addition and multiplication are $-\infty$ and 0 respectively.

The max-plus semifield is algebraically isomorphic to the *max-times semifield* $\mathbb{R}_{\max,\times}$, also known as the max-prod semifield (see e.g. [23,24]), which is given by the set $\mathbb{R}_+ = [0, +\infty)$ endowed with $\alpha \oplus \beta := \max(\alpha, \beta)$ as addition and the usual real numbers product $\alpha \otimes \beta := \alpha\beta$ as multiplication. Consequently, in the max-times semifield, 0 is the neutral element for addition and 1 is the neutral element for multiplication.

In this paper we consider both of these semifields at the same time, under the common notation $\mathbb{T}$ and under the common name *tropical algebra*. In what follows $\mathbb{T}$ denotes either the max-plus semifield $\mathbb{R}_{\max,+}$ or the max-times semifield $\mathbb{R}_{\max,\times}$. We will use $\mathbb{0}$ to denote the neutral element for addition, $\mathbb{1}$ to denote the neutral element for multiplication, and $\mathbb{T}_+$ to denote the set of all invertible elements with respect to the multiplication, i.e., all the elements of $\mathbb{T}$ different from $\mathbb{0}$.

The space $\mathbb{T}^n$ of n-dimensional vectors $x = (x_1, \ldots, x_n)$, endowed naturally with the componentwise addition (also denoted by $\oplus$) and $\lambda x := (\lambda \otimes x_1, \ldots, \lambda \otimes x_n)$ as the multiplication of a scalar $\lambda \in \mathbb{T}$ by a vector x, is a semimodule over $\mathbb{T}$. The vector $(\mathbb{0}, \ldots, \mathbb{0}) \in \mathbb{T}^n$ is also denoted by $\mathbb{0}$, and it is the identity for $\oplus$.

In *tropical convexity*, one first defines the *tropical segment* joining the points $x, y \in \mathbb{T}^n$ as the set $\{\alpha x \oplus \beta y \in \mathbb{T}^n \mid \alpha, \beta \in \mathbb{T},\ \alpha \oplus \beta = \mathbb{1}\}$, and then calls a set $\mathcal{C} \subseteq \mathbb{T}^n$ *tropically convex* if it contains the tropical segment joining any two of its points (see Fig. 1 below for an illustration of tropical segments in dimension 2). Similarly, the notions of *cone, halfspace, semispace, hemispace, convex hull, linear span, convex and linear combination*, can be transferred to the tropical setting (precise definitions are given below). Henceforth all these terms used without precisions should always be understood in the max-plus or max-times (i.e. tropical) sense.

The interest in this convexity (also known as *max-plus convexity* when $\mathbb{T} = \mathbb{R}_{\max,+}$, or *max-times convexity* or $\mathbb{B}$-*convexity* when $\mathbb{T} = \mathbb{R}_{\max,\times}$) comes from several fields, some of which we next review. Convexity in $\mathbb{T}^n$ and in more general semimodules was introduced by Zimmermann [29] under the name "extremal convexity" with applications e.g. to discrete optimization problems and it was studied by Maslov, Kolokoltsov, Litvinov, Shpiz and others as part of the Idempotent Analysis [17,19,22], inspired by the fact that the solutions of a Hamilton–Jacobi equation associated with a deterministic optimal control problem belong to structures similar to convex cones. Another motivation arises from the algebraic approach to discrete event systems initiated by Cohen et al. [6], since the reachable and observable spaces of certain timed discrete event systems are naturally equipped with structures of cones of $\mathbb{T}^n$ (see e.g. Cohen et al. [7]). Motivated by tropical algebraic geometry and applications in phylogenetic analysis, Develin and Sturmfels studied polyhedral convex sets in $\mathbb{T}^n$ thinking of them as classical polyhedral complexes [10].

Many results that are part of classical convexity theory can be carried over to the setting of $\mathbb{T}^n$: separation of convex sets and projection operators (Gaubert and Sergeev [14]), minimization of distance and description of sets of best approximation (Akian et al. [1]), discrete convexity results such as Minkowski theorem (Gaubert and Katz [11,12]), Helly, Carathéodory and Radon theorems (Briec and Horvath [2]), colorful Carathéodory and Tverberg theorems (Gaubert and Meunier [13]), to quote a few.

Here we investigate *hemispaces in* $\mathbb{T}^n$, which are convex sets in $\mathbb{T}^n$ whose complements in $\mathbb{T}^n$ are also convex. The definition of hemispaces makes sense in other structures once the notion of convex set is defined. Hemispaces also appear in the literature under the name of halfspaces, convex halfspaces, and generalized halfspaces. As general convex sets are quite complicated in many convexity structures, a simple description of hemispaces is highly desirable. Usual hemispaces in $\mathbb{R}^n$ are described by Lassak in [18]. Martínez-Legaz and Singer [20] give several geometric characterization of usual hemispaces in $\mathbb{R}^n$ with the aid of linear operators and lexicographic order in $\mathbb{R}^n$.

Hemispaces play a role in abstract convexity (see Singer [27], Van de Vel [28]), where they are used in the Kakutani Theorem to separate two convex sets from each other. The proof of Kakutani Theorem