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Line star sets for Laplacian eigenvalues

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ABSTRACT

Let G be a mixed graph with a nonzero Laplacian eigenvalue μ of multiplicity k. A line star set for μ in G is a set Y of k edges of G such that μ is not a Laplacian eigenvalue of $G - Y$. It is shown that line star set exists for any nonzero Laplacian eigenvalue of any mixed graph. Some basic properties for line star sets are given. We also obtain some results on line star sets in undirected graphs.

1. Introduction

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. For a simple undirected graph G, let A_G denote the $(0,1)$-adjacency matrix of G, and let D_G denote the diagonal matrix of vertex degrees of G. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. The eigenvalues of A_G, L_G and Q_G are called *A-eigenvalues*, *L-eigenvalues* and *Q-eigenvalues* of G, respectively. If μ is an A-eigenvalue of G with multiplicity k, then a *star set* for μ in G is a set $X \subseteq V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an A-eigenvalue. In this case, $G - X$ is called a *star complement* for μ in G. The research on star sets and star complements originated independently in papers by Ellingham [13] and Rowlinson [20].

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Let G be a mixed graph with the vertex set $V(G) = \{1, 2, \dots, n\}$. Some edges of G have orientation, while others do not (see [1,27]). For $(i, j) \in E(G)$, the sign of (i, j) is $\operatorname{sgn}(i, j) = 1$ if (i, j) is unoriented and $\operatorname{sgn}(i, j) = -1$ if (i, j) is oriented. Let $a_{ij} = \operatorname{sgn}(i, j)$ if $(i, j) \in E(G)$, and $a_{ij} = 0$ otherwise. Then $\mathcal{A}_G = (a_{ij})_{n\times n}$ is called the *adjacency matrix* of G. The degree of a vertex u in G is defined to be the number of all (oriented and unoriented) edges incident with u. The matrix $\mathcal{L}_G = \mathcal{D}_G + \mathcal{A}_G$ is the *Laplacian matrix* of G, where $\mathcal{D}_G$ is the diagonal matrix of vertex degrees of G (see [1,14]). The eigenvalues of $\mathcal{L}_G$ are called *Laplacian eigenvalues* of G. If all edges of G are unoriented, then $\mathcal{L}_G = Q_G$. If all edges of G are oriented, then $\mathcal{L}_G = L_H$, where H is the undirected graph obtained from G by deleting all orientation of G.

For a nonzero Laplacian eigenvalue μ of a mixed graph G with multiplicity k, we say that a set $Y \subseteq E(G)$ is a *line star set* for μ in G if $|Y| = k$ and μ is not a Laplacian eigenvalue of $G - Y$, where $G - Y$ denotes the spanning subgraph of G obtained by deleting all edges in Y. In this case, $G - Y$ is called a *line star complement* for μ in G. The concepts of line star set and line star complement are similar with those of star set and star complement.

This paper is organized as follows. In Section 2, some auxiliary lemmas are given. In Section 3, we give basic properties for line star sets in mixed graphs. In Section 4, we obtain some results on line star sets for L-eigenvalues. In Section 5, we obtain some results on line star sets for Q-eigenvalues. In Section 6, a comparison between line star sets and star sets is given.

2. Preliminaries

For a matrix A, if X is a matrix such that $AXA = A$, then X is called a $\{1\}$-*inverse* of A. If A is nonsingular, then A^{-1} is the unique $\{1\}$-inverse of A. If A is not nonsingular, then A has infinite $\{1\}$-inverses (see [3]). Let $A^{(1)}$ denote any $\{1\}$-inverse of A.

Lemma 2.1. *(See [3].) Let $A = PBQ$, where P, Q are nonsingular matrices. For any $\{1\}$-inverse $A^{(1)}$ of A, there exists a matrix C such that $A^{(1)} = Q^{-1}CP^{-1}$ and C is a $\{1\}$-inverse of B.*

Lemma 2.2. *(See [3].) The general solution of a solvable linear equations $Ax = b$ is $A^{(1)}b + (I - A^{(1)}A)u$, where u is an arbitrary column vector.*

Lemma 2.3. *(See [3].) For a matrix A, if A has full column rank, then $A^{(1)}A = I$. If A has full row rank, then $AA^{(1)} = I$.*

The resistance distance is a distance function on graphs introduced by Klein and Randić [17]. A connected undirected graph G can be viewed as an electrical network N by replacing each edge of G with a unit resistor. For any two vertices u, v in G, the *resistance distance* between them, denoted by $\Omega_G(u, v)$, is defined to be the effective resistance between them in the electrical network N (see [4,6,17]).

For a matrix M, let M_{uv} denote the (u, v)-entry of M.

Lemma 2.4. *(See [2].) Let G be a connected undirected graph. Then $\Omega_G(u, v) = M_{uu} + M_{vv} - M_{uv} - M_{vu}$, where M is any $\{1\}$-inverse of L_G.*

For two vertices u, v in an undirected graph G, let $d_G(u, v)$ denote the distance between them in G.

Lemma 2.5. *(See [17].) For any two vertices u, v in a tree T, we have $\Omega_T(u, v) = d_T(u, v)$.*

Lemma 2.6. *(See [15].) Let T be a tree with n vertices. If an integer $\mu > 1$ is an L-eigenvalue of T, then n is divisible by μ.*