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An invariance property for frameworks in Euclidean space

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ABSTRACT

We study frameworks in Euclidean space with a property of invariance with respect to similarity transformations. By methods of linear algebra, we address the problem of when a given graph can be realized as an invariant framework in a Euclidean space of dimension greater than or equal to three.

1. Introduction

Gauss' *Fundamental Theorem of Axonometry* affirms that the orthogonal projections into the complex plane of the three vertices of a cube adjacent to a vertex situated at the origin have sum of squares equal to zero. Conversely, any three complex numbers (not all zero) with sum of squares equal to zero arise in this way [6].

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Consider a cube placed in $\mathbb{R}^3$ and let $\varphi : \mathbb{R}^3 \to \mathbb{C}$ be any orthogonal projection onto the complex plane, for example $\varphi(x_1, x_2, x_3) = x_1 + \mathrm{i}x_2$. Let V be the collection of points in $\mathbb{R}^3$ corresponding to the vertex set of the cube. Then by Gauss' fundamental theorem, φ satisfies the equation

$$\sum_{y\sim x}\big(\varphi(y) - \varphi(x)\big)^2 = 0,$$

for each $x \in V$, where $y \sim x$ means that y is adjacent to x along an edge of the cube.

The above equation can be generalized to other regular polytopes [5]. For example, if $z_1, z_2, \dots, z_{N+1}$ are the orthogonal projections to $\mathbb{C}$ of the vertices of a regular simplex in $\mathbb{R}^N$, then

$$(z_1 + \cdots + z_{N+1})^2 = (N+1)\big({z_1}^2 + \cdots + {z_{N+1}}^2\big).$$

Thus if V is the vertex set of a regular simplex in $\mathbb{R}^N$ and φ is the function that associates to each element of V its value in the complex plane after an orthogonal projection, then φ satisfies the equation:

$$\frac{1}{N+1}\bigg(\sum_{y\sim x}\big(\varphi(y) - \varphi(x)\big)\bigg)^2 = \sum_{y\sim x}\big(\varphi(y) - \varphi(x)\big)^2.$$

The same applies to all the regular polytopes, where the factor $1/(N+1)$ is replaced by some other constant.

The above equation is invariant under any similarity transformation of the simplex, in particular, absolute position and size have no relevance. A question we can ask is to what extent the geometry of the simplex is contained in the combinatorial relationship between the vertices of the underlying graph and the coefficient $1/(N+1)$ that occurs on the left-hand side of the equation.

Given a graph $\Gamma = (V, E)$, with vertex set V and edge set E, together with a real-valued function $\gamma : V \to \mathbb{R}$, consider the equation:

$$\frac{\gamma(x)}{n(x)}\bigg(\sum_{y\sim x}\big(\varphi(y) - \varphi(x)\big)\bigg)^2 = \sum_{y\sim x}\big(\varphi(x) - \varphi(y)\big)^2, \tag{1}$$

at each vertex x, where $\varphi : V \to \mathbb{C}$ is a complex-valued function and $n(x)$ is the degree of Γ at x (the number of vertices adjacent to x). Solutions with $\gamma \equiv 0$ have been called *holomorphic functions*[2] and have been used to give a description of massless fields in a combinatorial setting [2]. Note that the equations are invariant by the replacement of φ by $\widetilde{\varphi} = \lambda\varphi + \mu$ ($\lambda, \mu \in \mathbb{C}$), as well as with respect to complex conjugation $\varphi \mapsto \overline{\varphi}$.

It is convenient to write $\Delta\varphi(x) = \frac{1}{n(x)}\sum_{y\sim x}(\varphi(y) - \varphi(x))$ (the Laplacian) and $(\nabla\varphi)^2(x) = \frac{1}{n(x)}\sum_{y\sim x}(\varphi(y) - \varphi(x))^2$ (the symmetric square of the derivative), whereby Eq. (1) becomes

$$\gamma(x)\Delta\varphi(x)^2 = (\nabla\varphi)^2(x). \tag{2}$$

A *framework* (or *body-bar framework*)[3] $\mathcal{F}$ in $\mathbb{R}^N$ is a pair (Γ, f), where $\Gamma = (V, E)$ is a finite simple graph (i.e. one without loops or multiple edges) with vertex set V and edge set E, together with a map $f : V \to \mathbb{R}^N$. We view the image of the vertices as a finite collection of points $\{\vec{x}_1, \dots, \vec{x}_n\}$ connected by edges which are straight line segments. By abuse of notation, we shall identify a vertex $x \in V$ with its image under f and write $\mathcal{F} = (V, E)$. We wish to realize a graph as a framework in Euclidean space so that the induced geometry is implicit in the combinatorial structure, rather than being dependent on the embedding.

[2] A notion of *holomorphic function* somewhat similar to this has been introduced by S. Barré [3]; however, in addition to (1) with $\gamma \equiv 0$, Barré requires that φ be harmonic.

[3] The term bar-joint framework is often used with the added condition that each bar has positive length. Body-bar frameworks are a special class of bar-joint frameworks.