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Nonexistence of exceptional 5-class association schemes with two Q-polynomial structures

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ABSTRACT

In Suzuki (1998) [7] Suzuki gave a classification of association schemes with multiple Q-polynomial structures, allowing for one exceptional case which has five classes. In this paper, we rule out the existence of this case. Hence Suzuki's theorem mirrors exactly the well-known counterpart for association schemes with multiple P-polynomial structures, a result due to Eiichi Bannai and Etsuko Bannai in 1980.

1. Introduction

Eiichi Bannai and Etsuko Bannai [2] studied association schemes with multiple P-polynomial structures and obtained the following classification. See [2] or [1, Section III.4].

Theorem 1. *Suppose that $\mathfrak{X}$ is a symmetric association scheme with a P-polynomial structure $A_0, A_1, \ldots, A_d$ for the adjacency matrices. If $\mathfrak{X}$ is not a polygon and has another P-polynomial structure, then the new structure is one of the following*:

(I) $A_0, A_2, A_4, A_6, \ldots, A_5, A_3, A_1$;
(II) $A_0, A_d, A_1, A_{d-1}, A_2, A_{d-2}, A_3, A_{d-3}, \ldots$;

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(III) $A_0, A_d, A_2, A_{d-2}, A_4, A_{d-4}, \ldots, A_{d-5}, A_5, A_{d-3}, A_3, A_{d-1}, A_1$;
(IV) $A_0, A_{d-1}, A_2, A_{d-3}, A_4, A_{d-5}, \ldots, A_5, A_{d-4}, A_3, A_{d-2}, A_1, A_d$.

Hence, $\mathcal{X}$ admits at most two P-polynomial structures.

The parametric conditions for each case above can be found in [1, Section III.4]. The question was raised whether similar result could be obtained for association schemes with multiple Q-polynomial structures in [1, Section III.4] and [2]. Suzuki [7] settled this question in the following theorem in 1998.

Theorem 2. *Suppose that $\mathcal{X}$ is a symmetric association scheme with a Q-polynomial structure $E_0, E_1, \ldots, E_d$ for the primitive idempotents. If $\mathcal{X}$ is not a polygon and has another Q-polynomial structure, then the new structure is one of the following*:

(I) $E_0, E_2, E_4, E_6, \ldots, E_5, E_3, E_1$;
(II) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \ldots$;
(III) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \ldots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$;
(IV) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \ldots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$;
(V) $d = 5$ *and* $E_0, E_5, E_3, E_2, E_4, E_1$.

Hence, $\mathcal{X}$ admits at most two Q-polynomial structures.

Note that case (V) has no counterpart in Theorem 1. In fact, its counterpart $A_0, A_5, A_3, A_2, A_4, A_1$ did appear in the original statement of Theorem 1 in [2] but was eliminated with an easy combinatorial argument. It has been wondered if this case can also be eliminated, e.g. [5, p. 1506]. We will do exactly this in this paper, and so Theorems 1 and 2 are dual to each other.

Theorem 3 *(Main theorem). Case* (V) *in Theorem* 2 *does not occur.*

We conclude this section with the outline of our proof. Any automorphism of the splitting field for an association scheme induces a permutation of its primitive idempotents. In particular, when applied to a given Q-polynomial structure, each non-trivial automorphism induces another Q-polynomial structure. By Theorem 2, the Galois group has order at most 2. If the Galois group has order 2 and fixes the Krein parameters, we can compare the Krein parameters from the two Q-polynomial structures and then determine the Krein matrix B_1^* for a putative association scheme $\mathcal{X}$ in case (V). It turns out that B_1^* is completely determined by the first multiplicity m. $\mathcal{X}$ has a fusion scheme with 3 classes, whose parameters can be obtained from B_1^*. Using elementary number theory we determine the possible values for m and further show that these values give infeasible parameters of the original scheme. If the Galois group is trivial or it has order 2 but does not fix the Krein parameters, we will derive two equations from the two Q-polynomial structures, which lead to a contradiction. All calculations are verified with the software package Maple 14.

All association schemes in this paper are symmetric. The reader is referred to [1,8] for missing definitions. For recent activities on P- or Q-polynomial association schemes and related topics, see the recent survey paper by Martin and Tanaka [5].

2. Preliminaries

In this paper, we adopt the notation in [1]. Let $\mathcal{X}$ be a symmetric association scheme with adjacency matrices A_i and primitive idempotents E_i, $0 \leqslant i \leqslant d$. Then $A_0, \ldots, A_d$ span an algebra $\mathcal{M}$ over the real field $\mathbb{R}$, called the Bose–Mesner algebra of $\mathcal{X}$.