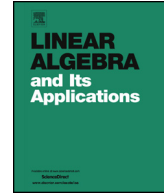




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Nonexistence of exceptional 5-class association schemes with two Q -polynomial structures



Jianmin Ma^{a,*}, Kaishun Wang^b

^a Hebei Key Lab. of Computational Mathematics & Applications and College of Math. & Info. Sciences, Hebei Normal University, Shijiazhuang 050016, China

^b Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China

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ABSTRACT

In Suzuki (1998) [7] Suzuki gave a classification of association schemes with multiple Q -polynomial structures, allowing for one exceptional case which has five classes. In this paper, we rule out the existence of this case. Hence Suzuki's theorem mirrors exactly the well-known counterpart for association schemes with multiple P -polynomial structures, a result due to Eiichi Bannai and Etsuko Bannai in 1980.

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1. Introduction

Eiichi Bannai and Etsuko Bannai [2] studied association schemes with multiple P -polynomial structures and obtained the following classification. See [2] or [1, Section III.4].

Theorem 1. *Suppose that \mathcal{X} is a symmetric association scheme with a P -polynomial structure A_0, A_1, \dots, A_d for the adjacency matrices. If \mathcal{X} is not a polygon and has another P -polynomial structure, then the new structure is one of the following:*

- (I) $A_0, A_2, A_4, A_6, \dots, A_5, A_3, A_1$;
- (II) $A_0, A_d, A_1, A_{d-1}, A_2, A_{d-2}, A_3, A_{d-3}, \dots$;

* Corresponding author.

E-mail addresses: jianminma@yahoo.com (J. Ma), wangks@bnu.edu.cn (K. Wang).

(III) $A_0, A_d, A_2, A_{d-2}, A_4, A_{d-4}, \dots, A_{d-5}, A_5, A_{d-3}, A_3, A_{d-1}, A_1$;

(IV) $A_0, A_{d-1}, A_2, A_{d-3}, A_4, A_{d-5}, \dots, A_5, A_{d-4}, A_3, A_{d-2}, A_1, A_d$.

Hence, \mathcal{X} admits at most two P -polynomial structures.

The parametric conditions for each case above can be found in [1, Section III.4]. The question was raised whether similar result could be obtained for association schemes with multiple Q -polynomial structures in [1, Section III.4] and [2]. Suzuki [7] settled this question in the following theorem in 1998.

Theorem 2. *Suppose that \mathcal{X} is a symmetric association scheme with a Q -polynomial structure E_0, E_1, \dots, E_d for the primitive idempotents. If \mathcal{X} is not a polygon and has another Q -polynomial structure, then the new structure is one of the following:*

(I) $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1$;

(II) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots$;

(III) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$;

(IV) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$;

(V) $d = 5$ and $E_0, E_5, E_3, E_2, E_4, E_1$.

Hence, \mathcal{X} admits at most two Q -polynomial structures.

Note that case (V) has no counterpart in Theorem 1. In fact, its counterpart $A_0, A_5, A_3, A_2, A_4, A_1$ did appear in the original statement of Theorem 1 in [2] but was eliminated with an easy combinatorial argument. It has been wondered if this case can also be eliminated, e.g. [5, p. 1506]. We will do exactly this in this paper, and so Theorems 1 and 2 are dual to each other.

Theorem 3 (Main theorem). *Case (V) in Theorem 2 does not occur.*

We conclude this section with the outline of our proof. Any automorphism of the splitting field for an association scheme induces a permutation of its primitive idempotents. In particular, when applied to a given Q -polynomial structure, each non-trivial automorphism induces another Q -polynomial structure. By Theorem 2, the Galois group has order at most 2. If the Galois group has order 2 and fixes the Krein parameters, we can compare the Krein parameters from the two Q -polynomial structures and then determine the Krein matrix B_1^* for a putative association scheme \mathcal{X} in case (V). It turns out that B_1^* is completely determined by the first multiplicity m . \mathcal{X} has a fusion scheme with 3 classes, whose parameters can be obtained from B_1^* . Using elementary number theory we determine the possible values for m and further show that these values give infeasible parameters of the original scheme. If the Galois group is trivial or it has order 2 but does not fix the Krein parameters, we will derive two equations from the two Q -polynomial structures, which lead to a contradiction. All calculations are verified with the software package Maple 14.

All association schemes in this paper are symmetric. The reader is referred to [1,8] for missing definitions. For recent activities on P - or Q -polynomial association schemes and related topics, see the recent survey paper by Martin and Tanaka [5].

2. Preliminaries

In this paper, we adopt the notation in [1]. Let \mathcal{X} be a symmetric association scheme with adjacency matrices A_i and primitive idempotents E_i , $0 \leq i \leq d$. Then A_0, \dots, A_d span an algebra \mathcal{M} over the real field \mathbb{R} , called the Bose–Mesner algebra of \mathcal{X} .

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