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# Structures and numerical ranges of power partial isometries

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ABSTRACT

We derive a matrix model, under unitary similarity, of an $n$-by-$n$ matrix $A$ such that $A, A^2, \dots, A^k$ ($k \geqslant 1$) are all partial isometries, which generalizes the known fact that if $A$ is a partial isometry, then it is unitarily similar to a matrix of the form $\begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix}$ with $B^*B + C^*C = I$. Using this model, we show that if $A$ has ascent $k$ and $A, A^2, \dots, A^{k-1}$ are partial isometries, then the numerical range $W(A)$ of $A$ is a circular disc centered at the origin if and only if $A$ is unitarily similar to a direct sum of Jordan blocks whose largest size is $k$. As an application, this yields that, for any $S_n$-matrix $A$, $W(A)$ (resp., $W(A \otimes A)$) is a circular disc centered at the origin if and only if $A$ is unitarily similar to the Jordan block $J_n$. Finally, examples are given to show that, for a general matrix $A$, the conditions that $W(A)$ and $W(A \otimes A)$ are circular discs at 0 are independent of each other.



## 1. Introduction

An $n$-by-$n$ complex matrix $A$ is a *partial isometry* if $\|Ax\| = \|x\|$ for any vector $x$ in the orthogonal complement $(\ker A)^\perp$ in $\mathbb{C}^n$ of the kernel of $A$, where $\|\cdot\|$ denotes the standard norm in $\mathbb{C}^n$. The study of such matrices or, more generally, such operators on a Hilbert space dates back to 1962 [7]. Their general properties have since been summarized in [6, Chapter 15].

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In this paper, we study matrices $A$ such that, for some $k \geqslant 1$, the powers $A, A^2, \dots, A^k$ are all partial isometries. In Section 2 below, we derive matrix models, under unitary similarity, of such a matrix (Theorems 2.2 and 2.4). They are generalizations of the known fact that $A$ is a partial isometry if and only if it is unitarily similar to a matrix of the form $\left[\begin{smallmatrix} 0 & B \\ 0 & C \end{smallmatrix}\right]$ with $B^*B + C^*C = I$ (Lemma 2.1).

Recall that the *ascent* of a matrix, denoted by $a(A)$, is the smallest integer $k \geqslant 0$ for which $\ker A^k = \ker A^{k+1}$. It is easily seen that $a(A)$ is equal to the size of the largest Jordan block associated with the eigenvalue 0 in the Jordan form of $A$. We denote the $n$-by-$n$ *Jordan block*

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

by $J_n$. The *numerical range* $W(A)$ of $A$ is the subset $\{\langle Ax, x\rangle \colon x \in \mathbb{C}^n,\ \|x\| = 1\}$ of the complex plane $\mathbb{C}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^n$. It is known that $W(A)$ is a nonempty compact convex subset, and $W(J_n) = \{z \in \mathbb{C}\colon |z| \leqslant \cos(\pi/(n+1))\}$ (cf. [5, Proposition 1]). For other properties of the numerical range, the readers may consult [6, Chapter 22] or [10, Chapter 1].

Using the matrix model for power partial isometries, we show that if $a(A) = k \geqslant 2$ and $A, A^2, \dots, A^{k-1}$ are all partial isometries, then the following are equivalent: (a) $W(A)$ is a circular disc centered at the origin, (b) $A$ is unitarily similar to a direct sum $J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_\ell}$ with $k = k_1 \geqslant k_2 \geqslant \cdots \geqslant k_\ell \geqslant 1$, and (c) $A$ has no unitary part and $A^j$ is a partial isometry for all $j \geqslant 1$ (Theorem 2.6). An example is given, which shows that the number “$k-1$” in the above assumption is sharp (Example 2.7).

In Section 3, we consider the class of $S_n$-matrices. Recall that an $n$-by-$n$ matrix $A$ is of *class* $S_n$ if $A$ is a contraction ($\|A\| \equiv \max\{\|Ax\|\colon x \in \mathbb{C}^n,\ \|x\| = 1\} \leqslant 1$), its eigenvalues are all in $\mathbb{D}$ ($\equiv \{z \in \mathbb{C}\colon |z| < 1\}$), and it satisfies $\operatorname{rank}(I_n - A^*A) = 1$. Such matrices are the finite-dimensional versions of the *compression of the shift* $S(\phi)$, first studied by Sarason [11]. They also feature prominently in the Sz.-Nagy–Foiaş contraction theory [12]. It turns out that a hitherto unnoticed property of such matrices is that if $A$ is of class $S_n$ and $k$ is its ascent, then $A, A^2, \dots, A^k$ are all partial isometries. Thus the structure theorems in Section 2 are applicable to $A$ or even to $A \otimes A$, the tensor product of $A$ with itself. As a consequence, we obtain that, for an $S_n$-matrix $A$, the numerical range $W(A)$ (resp., $W(A \otimes A)$) is a circular disc centered at the origin if and only if $A$ is unitarily similar to the Jordan block $J_n$ (Theorem 3.3). The assertion concerning $W(A)$ is known before (cf. [13, Lemma 5]). Finally, we give examples to show that if $A$ is a general matrix, then the conditions for the circularity (at the origin) of $W(A)$ and $W(A \otimes A)$ are independent of each other (Examples 3.5 and 3.6).

We use $I_n$ and $0_n$ to denote the $n$-by-$n$ identity and zero matrices, respectively. An identity or zero matrix with unspecified size is simply denoted by $I$ or 0. For an $n$-by-$n$ matrix $A$, $\operatorname{nullity} A$ is used for $\dim \ker A$, and $\operatorname{rank} A$ for its rank. The *real part* of $A$ is $\operatorname{Re} A = (A + A^*)/2$. The *geometric* and *algebraic multiplicities* of an eigenvalue $\lambda$ of $A$ are $\operatorname{nullity}(A - \lambda I_n)$ and the multiplicity of the zero $\lambda$ in the characteristic polynomial $\det(zI_n - A)$ of $A$, respectively. Note that $a(A)$ equals the smallest integer $k \geqslant 0$ for which the geometric and algebraic multiplicities of the eigenvalue 0 of $A^k$ coincide. An $n$-by-$n$ diagonal matrix with diagonal entries $a_1, \dots, a_n$ is denoted by $\operatorname{diag}(a_1, \dots, a_n)$.

## 2. Power partial isometries

We start with the following characterizations of partial isometries.

**Lemma 2.1.** *The following conditions are equivalent for an $n$-by-$n$ matrix $A$:*

(a) *$A$ is a partial isometry,*
(b) *$A^*A$ is an (orthogonal) projection, and*
(c) *$A$ is unitarily similar to a matrix of the form $\left[\begin{smallmatrix} 0 & B \\ 0 & C \end{smallmatrix}\right]$ with $B^*B + C^*C = I$.*

*In this case, $\left[\begin{smallmatrix} 0 & B \\ 0 & C \end{smallmatrix}\right]$ acts on $\mathbb{C}^n = \ker A \oplus (\ker A)^\perp$.*