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New upper bounds on the spectral radius of trees with the given number of vertices and maximum degree ☆

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ABSTRACT

This paper studies the problem of estimating the spectral radius of trees with the given number of vertices and maximum degree. We obtain the new upper bounds on the spectral radius of the trees, and the results are the best upper bounds expressed by the number of vertices and maximum degree, at present.

Let $T=(V,E)$ be a tree on n vertices with maximum degree Δ, where $3 \leqslant \Delta \leqslant n-2$. Denote by $\rho(T)$ the spectral radius of T. We prove that

(1) if $n \leqslant 2\Delta$, then $\rho(T) \leqslant \sqrt{\frac{n-1+\sqrt{(n-2\Delta)^2+2n-3}}{2}}$, and equality holds if and only if T is an almost completely full-degree tree of 3 levels;
(2) if $2\Delta < n \leqslant \Delta^2+1$, then $\rho(T) \leqslant \sqrt{2\Delta-1}$, and equality holds if and only if T is a completely full-degree tree of 3 levels;
(3) if $n > \Delta^2+1$, then $\rho(T) < 2\sqrt{\Delta-1}\cos\frac{\pi}{2k+1}$, where $k = \lceil \log_{\Delta-1}(\frac{(\Delta-2)(n-1)}{\Delta}+1)\rceil + 1$.

1. Introduction

Let $T=(V,E)$ be a tree on n vertices with maximum degree Δ. Its adjacency matrix is defined to be the $n \times n$ matrix $A(T)=(a_{ij})$, where $a_{ij}=1$ if v_i is adjacent to v_j; and $a_{ij}=0$, otherwise. The

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characteristic polynomial of T is $\det(xI - A(T))$ and is denoted by $\Phi(T, x)$. Since $A(T)$ is a symmetric matrix, each of its eigenvalues is real. We assume, without loss of generality, that they are ordered in nonincreasing order, i.e., $\rho_1(T) \geqslant \rho_2(T) \geqslant \cdots \geqslant \rho_n(T)$. We call them the eigenvalues of T. In particular, the largest eigenvalue $\rho_1(T)$ is called the spectral radius of T, denoted by $\rho(T)$. Since T is a connected graph, $A(T)$ is irreducible, the spectral radius is simple and there is a unique positive unit eigenvector by the Perron–Frobenius Theorem, e.g. [1]. We shall refer to such an eigenvector as the Perron vector of T.

Let d_v be the degree of v $(v \in V)$, $\Delta = \max\{d_v\colon\ v \in V\}$. Let $N_T(v)$ denote the set of vertices adjacent to v in T. A pendant vertex of T is a vertex of degree 1. Denote by $\rho(A)$ the largest eigenvalue of the matrix A. The distance $\mathrm{dist}(v, u)$ from a vertex v to a vertex u is the length of the shortest path joining v and u. We recall that the eccentricity e_u of a vertex u is the largest distance from u to any other vertex of the graph. We recall some results on the upper bounds of the spectral radius of trees.

In [2] Stevanović proves that

$$\rho(T) < 2\sqrt{\Delta - 1} \tag{1.1}$$

where T is a tree with maximum degree Δ.

In [3] Oscar Rojo gives an improved upper bound on the spectral radius of a tree. Let T be a tree with maximum degree Δ, u be a vertex of T such that $d_u = \Delta$. Let $k = e_u + 1$, where e_u is the eccentricity of u. For $j = 1, 2, \ldots, k - 1$, let $\delta_j = \max\{d_v\colon\ \mathrm{dist}(v, u) = j\}$. Then

$$\rho(T) < \max\Bigl\{\max_{2\leqslant j\leqslant k-2}\{\sqrt{\delta_j - 1} + \sqrt{\delta_{j-1} - 1}\}, \sqrt{\delta_1 - 1} + \sqrt{\Delta}\Bigr\}. \tag{1.2}$$

In [4] Oscar Rojo gives another upper bound on the spectral radius of a tree. Let T be a tree with maximum degree Δ and such that there exist two adjacent vertices u and v with $d_u = d_v = \Delta$. Let T' be the forest obtained from T by deleting the edge uv. Thus T' is the union of two disjoint trees $T_u = (V_u, E_u)$ and $T_v = (V_v, E_v)$. Let $k_u = e_u + 1$, $k_v = e_v + 1$, where e_u and e_v are the eccentricities of u and v with respect to the trees T_u and T_v respectively. Now, we define $k = \max\{k_u, k_v\}$, $\gamma_j(u) = \max\{d_x\colon\ x \in V_u,\ \mathrm{dist}(x, u) = j\}$, $1 \leqslant j \leqslant k - 2$, $\gamma_j(v) = \max\{d_y\colon\ y \in V_v,\ \mathrm{dist}(y, v) = j\}$, $1 \leqslant j \leqslant k - 2$, and $\gamma_j = \max\{\gamma_j(u), \gamma_j(v)\}$, $1 \leqslant j \leqslant k - 2$, where $\gamma_j(u) = 0$ for $j > k_u - 1$ or $\gamma_j(v) = 0$ for $j > k_v - 1$. Then

$$\rho(T) < \max\Bigl\{\max_{2\leqslant j\leqslant k-2}\{\sqrt{\gamma_j - 1} + \sqrt{\gamma_{j-1} - 1}\}, \sqrt{\gamma_1 - 1} + \sqrt{\Delta - 1}\Bigr\}. \tag{1.3}$$

This paper studies the problem of estimating the spectral radius of trees with the given number of vertices and maximum degree. When $\Delta = n - 1$ or $\Delta = 2$, the tree T is a star or a path, respectively, and it is easy to obtain the spectral radius of the trees. Thus in this paper we only study for $3 \leqslant \Delta \leqslant n - 2$. We obtain the new upper bounds on the spectral radius of trees, as the following two theorems. The results are the best upper bounds expressed by the number of vertices and maximum degree, at present.

Theorem 3.1. *Let $T = (V, E)$ be a tree on n vertices with maximum degree Δ, where $3 \leqslant \Delta \leqslant n - 2$. Denote by $\rho(T)$ the spectral radius of T. We prove that*

(1) *if $n \leqslant 2\Delta$, then $\rho(T) \leqslant \sqrt{\frac{n-1+\sqrt{(n-2\Delta)^2+2n-3}}{2}}$, and equality holds if and only if T is an almost completely full-degree tree of 3 levels;*
(2) *if $2\Delta < n \leqslant \Delta^2 + 1$, then $\rho(T) \leqslant \sqrt{2\Delta - 1}$, and equality holds if and only if T is a completely full-degree tree of 3 levels;*
(3) *if $n > \Delta^2 + 1$, then $\rho(T) < 2\sqrt{\Delta - 1}\cos\frac{\pi}{2k+1}$, where $k = \lceil \log_{\Delta-1}(\frac{(\Delta-2)(n-1)}{\Delta} + 1)\rceil + 1$.*

Theorem 3.2. *Let $T = (V, E)$ be a tree on n vertices with maximum degree Δ, where $\Delta = 3$ and $n \geqslant \Delta + 2$. Let t be the cardinality of the vertices of degree 3. We prove that*