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A classification of invariant distributions and convergence of imprecise Markov chains

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ABSTRACT

We analyse the structure of imprecise Markov chains and study their convergence by means of accessibility relations. We first identify the sets of states, so-called *minimal permanent classes*, that are the minimal sets capable of containing and preserving the whole probability mass of the chain. These classes generalise the *essential classes* known from the classical theory. We then define a class of extremal imprecise invariant distributions and show that they are uniquely determined by the values of the upper probability on minimal permanent classes. Moreover, we give conditions for unique convergence to these extremal invariant distributions.

1. Introduction

The term *imprecise Markov chain* denotes a Markov chain whose parameters are not fully determined, but rather some partial information about them is given in terms of models of *imprecise probabilities*. The systematic modelling of Markov chains with partially determined parameters was initiated by Hartfiel and Seneta (see e.g. [5,6] and the references therein) under the name 'Markov set chains'. Their theory is not formally built on the models of imprecise probabilities, such as Walley's model [11], although they develop similar concepts and methods. More formally the models of imprecise probabilities have been involved in the study of Markov chains by the approaches proposed by Škulj [9], where general models of interval probabilities (see e.g. [12]) are involved, and de Cooman,

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Hermans, and Quaeghebeur [4], who introduce the approach with upper and lower expectation functionals.

One of the main considerations of all mentioned approaches are the conditions for unique convergence to an invariant (imprecise) limit distribution. It has been shown that under certain assumptions imprecise Markov chains behave in a similar way as the precise ones, allowing the possibility to generalise concepts, such as accessibility relations [4], coefficients of ergodicity [5,10], and, for the chains with certain absorption, the invariant distributions conditional on non-absorption [3].

The main topic of the present paper is the imprecise Markov chains that do not converge uniquely, which means that the limit imprecise distribution depends on the initial state or distribution over the set of states. In the classical theory this case is relatively easily translated to the case with unique convergence by partitioning the set of states into disjoint *communication classes*, which are then either *essential* or *transient*. Essential classes have the property that the probability mass concentrated within them and in the transient classes leading to them is preserved within the class. In the long term it is then either distributed among states in a stationary way or moves between the states in *cycles*. Transient classes, though, in the long term transfer all the probability mass to the essential classes. The long term behaviour of a precise Markov chain is then described with the proportions of the probability mass within each essential class, that can then be examined, each separately without influencing the others, as either ergodic or cyclic single communication classes. (For more details see e.g. [8].)

The situation in the case of imprecise Markov chains is considerably more complex. In the precise case the probability mass can only move from one state into another, which is determined by strictly positive transition probability. In the imprecise case, though, it can happen that instead we have a probability interval that has a strictly positive upper and zero lower bound. Therefore, we have to take into account both scenarios – where a transition is possible and where it is not – within the same model (for better clarification see Example 1). This leads to numerous interesting new possibilities that are not observed in the precise theory. One immediate consequence is that there are different possible ways to define communication relations between states. *Say for instance, when the upper probability of transition from a state x to another state y is positive and the lower one zero, can we say that y is accessible from x or not*? We argue that questions of this kind cannot be answered with a single accessibility relation. Therefore, in addition to the accessibility relation defined by de Cooman et al. [4], that is based on upper probabilities and which we describe in Section 3.1, we define another accessibility relation in Section 3.2. This relation is defined between sets of states rather than between single states. Combining both relations will allow us to define minimal classes of states, called *minimal permanent classes*, that generalise essential communication classes from the classical theory in the sense that they are the minimal sets of states capable of possibly preserving the entire probability mass concentrated within them. However, unlike in the case of essential classes in the classical theory, this property does not prevent the possibility that the mass that can possibly be preserved within one class is entirely or partially transferred elsewhere, that is to other minimal permanent classes. Moreover, these classes are neither necessarily disjoint. In effect, a complex network of interdependence between the classes in possible in general. Our first main result in Theorem 2 shows that nevertheless all invariant imprecise distributions that possibly concentrate the entire probability mass in one or more minimal permanent classes are uniquely determined by the set of such classes. Further, our second main result in Theorem 3 shows that when probability mass is at least in the limit concentrated in these classes, the imprecise distributions over the set of states converge to the unique invariant distributions. These two theorems present the main results of this paper.

In addition to the main theorems, we believe that the analysis of the pair of accessibility relations contributes to better understanding of imprecise Markov chains in general, describing the complexity of their structure, and on the other hand, the induced minimal permanent classes seem to present the smallest irreducible units, counterparts to the essential communication classes in the classical Markov chains. Although, considerably more apparently irreducible complexity remains in the imprecise case due to complex interdependencies that may be present between the classes.

The paper has the following structure. In the next section we formally describe the model of imprecise Markov chains. In Section 3 we first describe the accessibility relation based on upper transitions, previously defined by de Cooman et al. [4], which here is referred as the weak accessibility