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On projections of arbitrary lattices ☆

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ABSTRACT

In this paper we prove that given any two point lattices $\Lambda_1 \subset \mathbb{R}^n$ and $\Lambda_2 \subset \mathbb{R}^{n-k}$, there is a set of k vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \Lambda_1$ such that Λ_2 is, up to similarity, arbitrarily close to the projection of Λ_1 onto the orthogonal complement of the subspace spanned by $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$. This result extends the main theorem of Sloane et al. (2011) [1] and has applications in communication theory.

It was recently proved [1] that any $(n-1)$-dimensional lattice can be approximated by a sequence of lattices such that each element is, up to similarity, the orthogonal projection of $\mathbb{Z}^n$ onto a hyperplane determined by a linear equation with integer coefficients. As a consequence of this fact, such projections can achieve packing densities arbitrarily close to the one of the best lattice packing in $\mathbb{R}^{n-1}$. A natural question that arises from this result is whether it still holds for other lattices than $\mathbb{Z}^n$. We give a positive answer to this question by showing that any $(n-k)$-dimensional lattice can be approximated by sequences of projections of *any* lattice in $\mathbb{R}^n$, generalizing the main theorem of [1]. The main result of this paper is the following:

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Theorem 1. *Let Λ_1 be an n-dimensional lattice and Λ_2 an $(n-k)$-dimensional lattice with Gram matrix A. Given $\varepsilon > 0$, there exists a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset \Lambda_1$, a Gram matrix A_V for Λ_V (the projection of Λ_1 onto the orthogonal complement of the subspace V spanned by the vectors $\boldsymbol{v}_i$) and a number c such that:*

$$\|A - cA_V\| < \varepsilon. \tag{1}$$

Let Λ be a lattice in $\mathbb{R}^n$. Theorem 1 implies, for instance, that the search for good $(n-k)$-dimensional lattice packings can be regarded as a search for vectors $\boldsymbol{v}_i$ in Λ such that the projection of Λ onto $V^\perp$ has good density. It is worth remarking that good lower bounds on the existence of dense projection lattices were derived in previous works (see [2] and [3]) through only geometric arguments. Furthermore, the approximation of an arbitrary lattice by a sequence of lattices with additional structure is a technique that has found useful applications in the context of sphere packings. For instance, dense subsets of lattices (in the sense of [4, p. 126]) were previously studied in [5, Chapter 1], [6, Chapter 4], [7], and are important for the establishment of the celebrated Minkowski–Hlawka lower bound on the existence of dense lattice packings [6, Chapter 4], [9, p. 14]. In a more general context, periodic packings are used to prove sharp bounds for the density of the best sphere packing (not necessarily a lattice packing) in [8].

Projection lattices naturally arise in the context of lattice packings. The densest packing in two dimensions, A_2, is equivalent to the projection of $\mathbb{Z}^n$ onto $(1, 1, 1)^\perp$ and, in general, A_n^* is the projection of $\mathbb{Z}^n$ onto $(1, \ldots, 1)^\perp$. Furthermore, the densest known packings in dimensions 6 and 7 (E_6 and E_7) can be defined as the intersection of the so-called Gosset lattice E_8 with certain hyperplanes determined by minimal vectors in E_8 [9], hence the duals E_7^* and E_6^* are exact projections of E_8. The problem of finding projections of $\mathbb{Z}^n$ with good packing density arises in the communication framework linked to error control for continuous alphabet sources, which is described in [2]. In [10], it is discussed how more general projections as presented here can be applied to this communication problem.

The proof of our main result, Theorem 1, is constructive and follows similar lines to the ones of [1]. For this proof, a characterization of primitive subsets in a lattice given in the next section is fundamental. The same characterization was recently used in [11] to make possible constructions of new record dense packings in some dimensions. The construction presented in Eq. (11) is a generalization of the construction in Section V of [3], what leads to a result for general lattices and projections onto subspaces of higher codimension, extending what is done for $\mathbb{Z}^n$ in [1]. Examples and further questions are presented in Sections 3 and 4.

1. Preliminaries

Let $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\}$ be a set of m linearly independent vectors in $\mathbb{R}^n$. A (point) *lattice* $\Lambda \subset \mathbb{R}^n$ with basis $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\}$ is defined as the set:

$$\Lambda = \{\alpha_1 \boldsymbol{g}_1 + \cdots + \alpha_m \boldsymbol{g}_m \mid \alpha_1, \ldots, \alpha_m \in \mathbb{Z}\}.$$

A matrix G whose rows are the basis vectors $\boldsymbol{g}_i$ is said to be a *generator matrix* for Λ. The matrix $A = GG^t$ is called a *Gram matrix* for Λ and the value $\det \Lambda = \det GG^t$ is the *determinant* or *discriminant* of Λ. Two matrices G and $\hat{G}$ generate the same lattice if there is a unimodular matrix U such that $G = U\hat{G}$. Although a lattice has infinitely many bases, the value $\det \Lambda$ is an invariant under change of basis. We say that a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset \Lambda$ is *primitive* if it can be extended to a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m$ of Λ. If $\boldsymbol{v}_i = \boldsymbol{a}_i G$, $\boldsymbol{a}_i \in \mathbb{Z}^m$, then a necessary and sufficient condition for a set of vectors to be primitive is that the gcd of the $k \times k$ minor determinants of the matrix $[\boldsymbol{a}_1^t, \boldsymbol{a}_2^t, \ldots, \boldsymbol{a}_k^t]$ equals ± 1 [4].

Two lattices are said *equivalent* if there exists a similarity transformation that takes on into another. Equivalently, two lattices with generator matrices G_1 and G_2 are equivalent if there exists an unimodular matrix U, an orthogonal matrix Q and a non-zero number c such that $G_1 = cUG_2Q$. Equivalent lattices have the same density, as well as other geometric properties (see [9] for undefined terms).