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Diagonals and numerical ranges of direct sums of matrices

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ABSTRACT

For any n-by-n matrix A, we consider the maximum number $k = k(A)$ for which there is a k-by-k compression of A with all its diagonal entries in the boundary $\partial W(A)$ of the numerical range $W(A)$ of A. If A is a normal or a quadratic matrix, then the exact value of $k(A)$ can be computed. For a matrix A of the form $B \oplus C$, we show that $k(A) = 2$ if and only if the numerical range of one summand, say, B is contained in the interior of the numerical range of the other summand C and $k(C) = 2$. For an irreducible matrix A, we can determine exactly when the value of $k(A)$ equals the size of A. These are then applied to determine $k(A)$ for a reducible matrix A of size 4 in terms of the shape of $W(A)$.

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1. Introduction

Let A be an n-by-n complex matrix. Its numerical range $W(A)$ is, by definition, the set $\{\langle Ax, x\rangle: \ x \in \mathbb{C}^n, \ \|x\| = 1\}$, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$, respectively. It is well known that $W(A)$ is a nonempty compact convex subset of the complex plane. For other properties of the numerical range, we refer the reader to [4, Chapter 1]. Let $k(A)$ be the maximum number k of orthonormal vectors $x_1, \dots, x_n \in \mathbb{C}^n$ with $\langle Ax_j, x_j\rangle$ in the boundary $\partial W(A)$ of $W(A)$ for all j. Note that $k(A)$ is also the maximum size of a compression of A with all its diagonal entries in $\partial W(A)$. Recall that a k-by-k matrix B is a *compression* of A if $B = V^*AV$ for some n-by-k matrix V with $V^*V = I_k$. In particular, if n equals k, then A and B are said to be *unitarily similar*, which we denote by $A \cong B$. The number $k(A)$ was introduced in [3] and [7]. It relates properties of the numerical range to the compressions of A. In particular, it was shown in

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[3, Lemma 4.1 and Theorem 4.4] that $2 \leqslant k(A) \leqslant n$ for any n-by-n ($n \geqslant 2$) matrix A, and $k(A) = \lceil n/2 \rceil$ for any S_n matrix A ($n \geqslant 3$). Recall that an n-by-n matrix A is of class S_n if it is a *contraction*, that is, $\|A\| \equiv \max_{\|x\|=1} \|Ax\| \leqslant 1$, its eigenvalues are all in the open unit disc $\mathbb{D} \equiv \{z \in \mathbb{C}\colon\ |z| \leqslant 1\}$, and the rank of $I_n - A^*A$ equals one. In [7, Theorem 3.1], it was proven that $k(A) = n$ for an n-by-n ($n \geqslant 2$) weighted shift matrix A with weights $w_1, \ldots, w_n$ if and only if either $|w_1| = \cdots = |w_n|$ or n is even and $|w_1| = |w_3| = \cdots = |w_{n-1}|$ and $|w_2| = |w_4| = \cdots = |w_n|$. Recall that an n-by-n ($n \geqslant 2$) matrix of the form

$$\begin{bmatrix} 0 & w_1 & & \\ & 0 & \ddots & \\ & & \ddots & w_{n-1} \\ w_n & & & 0 \end{bmatrix}$$

is called a *weighted shift matrix* with *weights* $w_1, \ldots, w_n$.

In Section 2 below, we first determine the value of $k(A)$ for a normal matrix A (Proposition 2.1). Then we consider the direct sum $A = B \oplus C$, where the numerical ranges $W(B)$ and $W(C)$ are assumed to be disjoint. In this case, we show that the value of $k(A)$ is equal to the sum of $k_1(B)$ and $k_1(C)$ (Theorem 2.2), where $k_1(B)$ and $k_1(C)$ are defined as follows. We define $k_1(B)$ to be the maximum number k for which there are orthonormal vectors $x_1, \ldots, x_k$ in $\mathbb{C}^n$ such that $\langle Bx_i, x_i \rangle$ is in $\partial W(A) \cap \partial W(B)$ for all $i = 1, \ldots, k$, and similarly for $k_1(C)$. Based on the proof of Theorem 2.2, we obtain the same formula for $k(A)$ under a slightly weaker condition on B and C (Theorem 2.4). In Section 3, we give some applications of Theorem 2.4. The first one (Proposition 3.1) shows that the equality $k(A) = k_1(B) + k_1(C)$ holds for a matrix A of the form $B \oplus C$ with normal C. In particular, we are able to determine the value of $k(A)$ for any 4-by-4 reducible matrix A (Corollary 3.4 and Propositions 3.7–3.9). Moreover, the number $k(A \oplus (A + aI_n))$ can be determined for any n-by-n matrix A and nonzero complex number a (Proposition 3.10). At the end of Section 3, we propose several open questions on $k(B \oplus C)$ and give a partial answer for one of them (Proposition 3.11). That is, the equality $k(\bigoplus_{j=1}^{m} A) = m \cdot k(A)$ holds if the dimension of $H_\xi(A)$ equals one for each $\xi \in \partial W(A)$, where the subspace $H_\xi(A)$ is defined in the first paragraph of Section 2. By using this, we can determine the value of $k(A)$ for a quadratic matrix A (Corollary 3.12). Recall that a *quadratic matrix* A is one which satisfies $A^2 + z_1 A + z_2 I = 0$ for some scalars z_1 and z_2.

We end this section by fixing some notation. For any finite square matrix A, we use $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/(2i)$ to denote its *real* and *imaginary parts*, respectively. The set of eigenvalues of A is denoted by $\sigma(A)$. A is called *positive definite*, denoted by $A > 0$, if A is Hermitian and $\langle Ax, x \rangle > 0$ for all $x \neq 0$. I_n is the n-by-n identity matrix. The n-by-n diagonal matrix with diagonals $\xi_1, \ldots, \xi_n$ is denoted by $\operatorname{diag}(\xi_1, \ldots, \xi_n)$. The *cardinal number* of a set S is $\#(S)$. The notation δ_{ij} is the *Kronecker delta*, i.e., δ_{ij} has the value 1 if $i = j$, and the value 0 if otherwise. The *span* of a nonempty subset S of a vector space V, denoted by $\operatorname{span}(S)$, is the subspace consisting of all linear combinations of the vectors in S.

2. Direct sum

We start by reviewing a few basic facts concerning the boundary points of a numerical range. For an n-by-n matrix A, a point ξ in $\partial W(A)$ and a supporting line L of $W(A)$ which passes through ξ, there is a θ in $[0, 2\pi)$ such that the ray from the origin which forms angle θ from the positive x-axis is perpendicular to L. In this case, $\operatorname{Re}(e^{-i\theta}\xi)$ is the maximum eigenvalue of $\operatorname{Re}(e^{-i\theta}A)$ with the corresponding eigenspace $E_{\xi,L}(A) \equiv \ker \operatorname{Re}(e^{-i\theta}(A - \xi I_n))$. Let $K_\xi(A)$ denote the set $\{x \in \mathbb{C}^n\colon\ \langle Ax, x \rangle = \xi \|x\|^2\}$ and $H_\xi(A)$ the subspace spanned by $K_\xi(A)$. If the matrix A is clear from the context, we will abbreviate these to $E_{\xi,L}$, K_ξ and H_ξ, respectively. For other related properties, we refer the reader to [2, Theorem 1] and [7, Proposition 2.2]. The next proposition on the value of $k(A)$ for a normal matrix A is an easy consequence of [7, Lemma 2.9]. It can be regarded as a motivation for our study of this topic.

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