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Parity successions in set partitions

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ABSTRACT

Suppose that the elements within each block of a partition π of $[n] = \{1, 2, \dots, n\}$ are written in ascending order. By a *parity succession*, we will mean a pair of adjacent elements x and y within some block of π such that $x \equiv y \pmod 2$. Here, we consider the problem of counting the partitions of $[n]$ according to the number of successions, extending recent results concerning successions on subsets and permutations. Using linear algebra, we determine a formula for the generating function which counts partitions having a fixed number of blocks according to size and number of successions. Furthermore, a special case of our formula yields an explicit recurrence for the generating function which counts the *parity-alternating* partitions of $[n]$, i.e., those that contain no successions.

1. Introduction

By a *partition* of a finite set, we will mean a collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is the set. If n and m are positive integers, then let us denote the set of all partitions of $[n] = \{1, 2, \dots, n\}$ having m blocks by $P_{n,m}$. Recall that the cardinality of $P_{n,m}$ is given by the Stirling number of the second kind, denoted hereafter as $S(n, m)$.

In what follows, we will write the elements within each block of $\pi \in P_{n,m}$ in increasing order. By a *parity succession*, we will mean a pair of adjacent elements x and y within some block of π such that $x \equiv y \pmod 2$. For example, if $\pi = 1356/2478/9 \in P_{9,3}$, then π has three parity successions (two in the first block with $1, 3$ and $3, 5$ and one in the second block with $2, 4$). We will refer to a partition which contains no successions as *parity-alternating*, that is, the elements within each block alternate

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between even and odd values. This concept of parity succession for set partitions extends an earlier one that was introduced for subsets [5] and later considered on permutations [6]. The terminology is an adaptation of an analogous usage in the study of integer sequences $(p_1, p_2, \ldots)$ in which a succession refers to a pair p_i, p_{i+1} with $p_{i+1} = p_i + 1$ (see, e.g., [1,9,2]).

Enumeration of discrete structures according to the parity of individual elements started perhaps with the following formula obtained by Tanny [10] for the number $g(n,k)$ of alternating k-subsets of $[n]$ given by

$$g(n,k) = \binom{\lfloor \frac{n+k}{2} \rfloor}{k} + \binom{\lfloor \frac{n+k-1}{2} \rfloor}{k}.$$

This result was later extended to any modulus in [3] and in terms of counting by successions in [5]. Tanimoto [8] considered the comparable problem on permutations in his investigation of signed Eulerian numbers and provided the formula for the number $h(n)$ of parity-alternating permutations of length n given by

$$h(n) = \frac{3 + (-1)^n}{2} \left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n+1}{2} \right\rfloor!.$$

This was later generalized in terms of succession counting in [6]; see also [4].

In this paper, we consider the problem of counting finite set partitions according to the number of successions, as defined above. We first consider a new two-parameter generalization, denoted by $S(n,m,a,b)$, of the Stirling numbers (see, e.g., [12] for other examples), which counts the members of $P_{n,m}$ in terms of the number of successions and another statistic. Using linear algebra, we find a formula for the bivariate generating function of this array in the first and last parameters where the middle two are fixed (see Theorem 7 below).

To do so, we first write all of the recurrences for the first m positive integers which are satisfied by this generating function in a block matrix form. We then find a recurrence satisfied by the blocks in a column of the inverse of a certain tridiagonal block matrix to complete the derivation, after observing that the inverse exists. Our formula shows, in fact, that the aforementioned generating function is always rational. In the third section, we particularize our formula to obtain explicit recurrences for generating functions which count the number of partitions having a fixed number of blocks and no successions. Using these recurrences, some enumerative results for parity-alternating set partitions are obtained, which may then be explained combinatorially.

2. General formulas

In order to count the partitions of $[n]$ according to the number of successions, we introduce a second statistic on $P_{n,m}$ which counts the number of blocks within a partition having an odd largest element. Let $P_{n,m,a,b}$ denote the subset of $P_{n,m}$ whose members contain exactly a blocks in which the largest element is odd and b successions. Let $S(n,m,a,b) = |P_{n,m,a,b}|$. For example, if $n = 5$ and $m = 3$, then we have $S(5,3,2,2) = 3$, the enumerated partitions being 13/24/5, 15/24/3, and 1/24/35, while $S(5,3,1,1) = 4$, the partitions in this case being 15/2/34, 12/35/4, 134/2/5, and 14/2/35. Note that $S(n,m) = \sum_{a,b} S(n,m,a,b)$. For example, when $n = 4$ and $m = 3$, we have

$$S(4,3) = S(4,3,1,0) + S(4,3,1,1) + S(4,3,2,0) + S(4,3,2,1) = 3 + 1 + 1 + 1 = 6.$$

The array $S(n,m,a,b)$ is determined as follows.

Proposition 1. *The array $S(n,m,a,b)$ may assume non-zero values only when $n \geqslant 0$, $0 \leqslant m \leqslant n$, $0 \leqslant a \leqslant m$, and $0 \leqslant b \leqslant n-m$. When $m \geqslant 1$ and $n \geqslant \frac{m-1}{2}$, it satisfies the recurrences given by*

$$S(2n,m,a,b) = S(2n-1,m-1,a,b) + (m-a)S(2n-1,m,a,b-1) \\ + (a+1)S(2n-1,m,a+1,b) \tag{1}$$

and