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The classification of Leonard triples that have Bannai/Ito type and odd diameter

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ABSTRACT

Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard triple* on V we mean an ordered triple of linear transformations A, A^*, A^ε in $\mathrm{End}(V)$ such that for each $B \in \{A, A^*, A^\varepsilon\}$ there exists a basis for V with respect to which the matrix representing B is diagonal and the matrices representing the other two linear transformations are irreducible tridiagonal. The *diameter* of the Leonard triple (A, A^*, A^ε) is defined to be one less than the dimension of V. In this paper we define a family of Leonard triples said to have Bannai/Ito type. We classify up to isomorphism the Leonard triples that have Bannai/Ito type and odd diameter. We show that each of these Leonard triples satisfies the $\mathbb{Z}_3$-symmetric Askey–Wilson relations.

1. Introduction

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field of characteristic zero.

We now recall the notions of a Leonard pair and a Leonard triple. To do this we use the following terms. Let X denote a square matrix. Then X is called *upper* (resp. *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal. X is called *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever all entries immediately above and below the main diagonal are nonzero.

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Definition 1.1. (See [11, Definition 1.1].) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

The *diameter* of the Leonard pair (A, A^*) is defined to be one less than the dimension of V.

Definition 1.2. (See [4, Definition 1.2].) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard triple* on V, we mean an ordered triple of linear transformations $A : V \to V$, $A^* : V \to V$ and $A^\varepsilon : V \to V$ that satisfy the conditions (i)–(iii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ε are irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A and A^ε are irreducible tridiagonal.
(iii) There exists a basis for V with respect to which the matrix representing A^ε is diagonal and the matrices representing A and A^* are irreducible tridiagonal.

The *diameter* of the Leonard triple (A, A^*, A^ε) is defined to be one less than the dimension of V.

For any Leonard triple, any two of the three form a Leonard pair. We say these Leonard pairs are *associated* with the Leonard triple.

Leonard pairs were introduced by Terwilliger [11] to extend the algebraic approach of Bannai and Ito [1] to a result of D. Leonard concerning the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal polynomials. Terwilliger classified the Leonard pairs up to isomorphism in [13]. By that classification, when $\mathbb{K}$ is a field of characteristic zero, the isomorphism classes of Leonard pairs fall naturally into twelve families: q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, affine q-Krawtchouk, quantum q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk and Bannai/Ito.

Leonard triples were introduced by Curtin in [4]. Leonard triples are related to spin models [3], the generalized Markov problem in number theory and the Poncelet problem in projective geometry [9]. It is important to classify all Leonard triples up to isomorphism. There are some beautiful results for the classification of Leonard triples. Curtin classified up to isomorphism the modular Leonard triples in [4]. Brown classified up to isomorphism the totally bipartite and the totally almost bipartite Leonard triples of Bannai/Ito type in [2]. Huang classified up to isomorphism the Leonard triples of q-Racah type in [8]. Gao, Wang and Hou classified up to isomorphism the Leonard triples of Racah type in [6]. Motivated by [6,8], Hou, Xu and Gao defined a family of Leonard triples said to have Bannai/Ito type and classified up to isomorphism these Leonard triples with even diameters in [7]. In this paper we continue to consider the Leonard triples of Bannai/Ito type and classify up to isomorphism these Leonard triples with odd diameters. We remark that we have to discuss separately two cases of even and odd diameters because the techniques used in [7] differ from those of the present paper.

The present paper is organized as follows. In Sections 2–5 we recall some basic concepts and results concerning Leonard pairs and Leonard triples. In Section 6 we discuss a certain sequence $\theta_i = (-1)^i(i + u - \frac{d}{2})$ $(0 \leqslant i \leqslant d)$. In Section 7 we define a family of Leonard systems said to have Bannai/Ito type and discuss some related concepts. In Sections 8–10 we define a family of Leonard pairs said to have Bannai/Ito type. For a given Leonard pair (A, A^*) that has Bannai/Ito type and odd diameter, we show that there exists a unique $A^\varepsilon \in \mathrm{End}(V)$ such that A, A^*, A^ε satisfy the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. We also find necessary and sufficient conditions for the triple (A, A^*, A^ε) to be a Leonard triple of Bannai/Ito type. In Section 11 we classify up to isomorphism the Leonard triples