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Linear flows and Morse graphs: Topological consequences in low dimensions

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ABSTRACT

The main aim of this paper is to show some specific connections between linear dynamic and graphs. Precisely, the Morse decomposition of a linear flow on the Grassmannians induces a directed graph. We apply the results appearing in Ayala et al. (2006, 2005) [2,3] and Colonius et al. (2002) [4] and compute the associated graphs for linear flows in dimensions two and three.

1. Introduction

The main aim of this paper is to show some connections between linear dynamic and graphs. Precisely, a linear flow Φ on $\mathbb{R}^d$ induces a well defined non-linear flow on the projective space and on every level $\mathbb{G}_i = \mathrm{Grass}(i, d)$: the Grassmannian manifold of the i-planes on $\mathbb{R}^d$, for $i \geqslant 1$. In order to analyze topological equivalence of linear flows in [3] and in a more general set up: the topological equivalence of bilinear control systems in [4], the authors introduce a directed graph $Gr(\Phi)$ associated with Φ. On the other hand, they also consider some spectral objects through the definition of the Lyapunov normal form, the short Lyapunov normal form and the zero short Lyapunov normal form of Φ. In particular, in the hyperbolic case, i.e., when Φ_1 and Φ_2 are hyperbolic, it is possible to

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characterize their dynamic. In fact, Φ_1 and Φ_2 are topological equivalent if and only if the dimensions of the stable (or unstable) subspaces are the same. This is something that can be seen from the short-zero Lyapunov form.

For representative linear flows in the plane and in the space we explicitly compute in this paper their dynamics on the Grassmannians and the corresponding associated graphs. In particular, it is possible in both dimensions to analyze the topological equivalence of a couple of linear hyperbolic dynamics just by looking at the graphs. Furthermore, in [3] it is proved that two arbitrary matrices in $gl(d,\mathbb{R})$ have isomorphic associated graphs if and only if their short Lyapunov forms are equal.

The article is organized as follows: Section 2 contains the basic notion of flows, Morse decomposition, the Lyapunov normal form, the Short Lyapunov normal form and the chain transitive set. Section 3 includes the Selgrade theorem [6], its generalization to the Grassmannian manifolds [3,4] and the construction of the direct graph associated to any linear differential equation. Finally, Section 4 contains the analysis on dimension two and three. We also include some examples.

2. Preliminaries

In this section the main definitions and results motivating this article are presented. We follow Refs. [2,3] and [4]. For a flow Φ on a compact metric space Y, a compact subset $K \subset Y$ is said to be *isolated invariant* if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int}(N)$ such that $\Phi(t,x) \in N$, $\forall t \in \mathbb{R}$ implies $x \in K$.

By using $\Phi(t,x)$ and $t \in \mathbb{R}$, a point $x \in Y$ can be moved toward the future or past. The *ω-limit* and *α-limit sets* of x are respectively defined as

$$\omega(x) = \Big\{ y \in Y \colon \exists t_n \to \infty \colon \lim_{n\to\infty} \Phi(t_n, x) = y \Big\},$$
$$\alpha(x) = \Big\{ y \in Y \colon \exists t_n \to -\infty \colon \lim_{n\to\infty} \Phi(t_n, x) = y \Big\}.$$

Definition 1. A *Morse decomposition* $\mathfrak{M}$ of a flow Φ on a compact metric space Y is a finite collection $\{\mathcal{M}_i \colon i = 1,\dots,n\}$ of non-empty, pairwise disjoint and isolated compact invariant sets such that

(i) $\forall x \in Y \colon \omega(x), \alpha(x) \subset \bigcup_{i=1}^n \mathcal{M}_i$.
(ii) Suppose that there are $\mathcal{M}_{j_0},\dots,\mathcal{M}_{j_l}$ and $x_1,\dots,x_l \in Y \setminus \bigcup_{i=1}^n \mathcal{M}_i$ such that $\alpha(x_k) \in \mathcal{M}_{j_{k-1}}$ and $\omega(x_k) \in \mathcal{M}_{j_k}$ for $k = 1,\dots,l$, then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The property (ii) implies that cycles are not allowed. The elements of a Morse decomposition are called *Morse sets*. Furthermore, an order is defined on a Morse decomposition in the following form:

For $i, i' \in \{1,\dots,n\}$: $\mathcal{M}_i \preceq \mathcal{M}_{i'}$ if there exists $x \in Y$ with $\alpha(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_{i'}$.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be Morse decompositions on a compact metric space Y. Then, $\mathfrak{M}_2$ is finer than $\mathfrak{M}_1$, if every element of $\mathfrak{M}_2$ is contained in some element of $\mathfrak{M}_1$.

Definition 2. Consider a flow Φ on a compact metric space (Y,δ), where δ is a metric. Let ε, $T > 0$. A *(ε,T)-chain* from $x \in Y$ to $y \in Y$ is given by a natural number $n \in \mathbb{N}$, together with points $x = x_0,\dots,x_n = y$ in Y, and times $T_0,\dots,T_{n-1} > T$, such that $\delta(\Phi(T_i,x_i),x_{i+1}) < \varepsilon$, for $i = 0,1,\dots,n-1$.

Definition 3. A subset $A \subset Y$ is *chain transitive* if for all $x, y \in A$ and ε, $T > 0$ there exists an (ε,T)-chain from x to y. A point $x \in Y$ is *chain recurrent* if for all ε, $T > 0$ there exists an (ε,T)-chain from x to x. The *chain recurrent set* $R \subset Y$ is the set of all chain recurrent points.

Theorem 4. *(See [2].) The connected components of the chain recurrent set R coincide with the maximal (with respect to set inclusion) chain transitive subsets of R. Furthermore, the flow restricted to a connected component of R is chain transitive.*

It is well known that except for the order of the blocks, any matrix A on a real vector space can be represented just by using its real Jordan form (RJF), $J(A)$ see [5]. Thus, in this article only the RJF is used to represent the corresponding system of differential equations $\dot{x} = Ax$.