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Generalized joint spectral radius and stability of switching systems

M. Ogura *, C.F. Martin

Department of Mathematics and Statistics, Texas Tech University, Broadway and Boston, Lubbock, TX 79409-1042, USA

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ABSTRACT

This paper extends the notion of generalized joint spectral radius with exponents, originally defined for a finite set of matrices, to probability distributions. We show that, under a certain invariance condition, the radius is calculated as the spectral radius of a matrix that can be easily computed, extending the classical counterpart. Using this result we investigate the mean stability of switching systems. In particular we establish the equivalence of mean square stability, simultaneous contractibility in square mean, and the existence of a quadratic Lyapunov function. Also the stabilization of positive switching systems is studied. Numerical examples are given to illustrate the results.

1. Introduction

Let $\mathcal{M}$ be a set of m square real matrices. The *generalized joint spectral radius* [1] of $\mathcal{M}$ with exponent $p > 0$ is defined by

$$\rho_p := \lim_{k\to\infty} \left(m^{-k} \sum_{A_1,\ldots,A_k\in\mathcal{M}} \|A_k \cdots A_1\|^p \right)^{1/kp} \tag{1}$$

* Corresponding author.
E-mail addresses: masaki.ogura@ttu.edu (M. Ogura), clyde.f.martin@ttu.edu (C.F. Martin).

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where $\|\cdot\|$ is any matrix norm. This quantity, firstly introduced [2,3] for $p = 1$ and then extended [1] for general p, has found many applications including wavelet theory, approximation theory, and fractal theory (for detail see [4] and references therein).

Among its applications is the stability theory of switching systems. Switching systems have been extensively studied in system and control theory literature due to their wide applicability. The paper [5] gives a detailed overview of the stability theory of switching systems. One of the most basic models of such systems is written by the difference equation

$$\Sigma_{\mathcal{M}}\colon\ x(k+1) = A_k x(k), \quad A_k \in \mathcal{M}. \tag{2}$$

Recently Jungers et al. [6] pointed out that the generalized joint spectral radius (1) can be used to study the so-called *pth mean stability* [7] of $\Sigma_{\mathcal{M}}$, which requires that the pth power of the norm of the trajectory $x(k)$, in average, converges to 0 as $k \to \infty$.

However, in several cases, we may know a priori not only the set $\mathcal{M}$ of possible values of the matrix A_k but also the *frequency* at which each matrix in $\mathcal{M}$ arises [7]. Such frequency can be expressed as a probability distribution μ on the set $\mathcal{M}$, possibly infinite, and this consideration leads us to the following *stochastic* switching system:

$$\Sigma_{\mu}\colon\ x(k+1) = A_k x(k), \quad A_k \text{ follows } \mu \text{ independently}. \tag{3}$$

Then the generalized joint spectral radius (1) fails to study the mean stability of Σ_μ because the mean stability takes the average of the norm of $x(k)$ with the weight determined by μ, while the definition (1) leads us to use the equal weight regardless of μ.

Motivated by this observation, in this paper we extend the notion of "classical" generalized joint spectral radius (1) to *probability distributions* so that it appropriately takes such weights into account (for precise definition see Definition 2.1). Then we actually show that the extended version of the generalized joint spectral radius characterizes the mean stability of the stochastic switching system Σ_μ. Our characterization generalizes several stability criteria that have been obtained in system theory literature.

Also we will show that, under a certain invariance condition on the probability measure, the radius admits an expression as the spectral radius of an easily computable matrix, which generalizes the classical counterparts [1,8,9]. The class of switching systems induced by the probability distributions satisfying the condition includes so-called positive switching systems [10–12].

Furthermore, using the characterization of mean stability, we will investigate the second mean stability, which is the special case of the pth mean stability. We will establish the equivalence between the following three different stability notions; the second mean stability, the existence of a quadratic Lyapunov function, and the stochastic version of the so-called simultaneous contractibility. We can regard this result as a stochastic variant of the result given in [13], where the authors studied the relationship between the existence of a quadratic Lyapunov function and the quantity called joint spectral radius [14] that is known [15] to determine the stability of the switching system (2). This result will be illustrated using a switching system taken from [16].

This paper is organized as follows. After preparing necessary mathematical notation and conventions, in Section 2 we give an extension of the classical generalized joint spectral radius with exponents and then state the main result. Section 3 gives an overview of the mean and moment stability of stochastic switching systems. The proof of the main result is given in Section 4. Using the main result, in Section 5 we give characterizations of the mean stability of stochastic switching systems. Finally in Section 6 we study the stabilization of positive stochastic switching systems.

1.1. Mathematical preliminaries

Let $\mathbb{R}$ and $\mathbb{Q}$ denote the set of real and rational numbers, respectively. Let $\mathbb{R}_+ := \{\alpha \in \mathbb{R}\colon\ \alpha \geqslant 0\}$. For a set X its cardinality is denoted by $|X|$. A real sequence $\{\alpha_k\}_{k=1}^{\infty}$ is said to be subadditive if for every k and ℓ we have $\alpha_{k+\ell} \leqslant \alpha_k + \alpha_\ell$. Fekete's subadditive lemma [17, Ch. 3, Sect. 1] states that any subadditive sequence $\{\alpha_k\}_{k=1}^{\infty}$ has a limit in $[-\infty, \infty)$ and the limit coincides with $\inf_{k \geqslant 1} \alpha_k$. We will need the next variant of this lemma.