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Distance spectral radius of trees with fixed number of pendent vertices [☆]

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The distance spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of the distance matrix $D(G)$. In this paper, we characterize the graph with minimum distance spectral radius among trees with fixed number of pendent vertices.

1. Introduction

In this paper, a graph means a simple connected undirected graph. Let $G = (V, E)$ be a graph with n vertices. For two vertices v_i and v_j $(i \neq j)$ in G, the *distance* between v_i and v_j, denoted by d_{ij} or $d(v_i, v_j)$, is the length of a shortest path joining v_i and v_j. The *distance matrix* of G is the matrix $D(G) = (d_{ij})$. It is obvious that $D(G)$ is a real symmetric matrix. Thus its eigenvalues are real numbers. We denote by $\rho(G)$ the largest eigenvalue of $D(G)$, and call it *the distance spectral radius* of G. By the Perron–Frobenius theorem, we know that $\rho(G) > 0$ and there exists a unique positive

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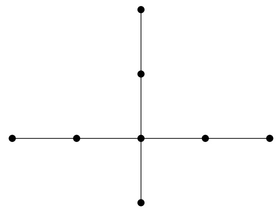

Fig. 1. $T_{\min}(8,4)$.

unit vector $x = (x_1, x_2, \ldots, x_n)^T$ called *Perron eigenvector* such that $D(G)x = \rho(G)x$, where x_i is the coordinate for the vertex v_i.

The distance matrix of a graph came up in many areas, including communication network design [6], graph embedding theory [4,7,8], molecular stability [11,16] and network flow algorithms [3,5]. Balaban et al. [1] proposed the use of $\rho(G)$ as a molecular descriptor, while in [10] it was used to infer the extent of branching and model boiling points of alkanes. So it is of interest to study the distance matrix and its eigenvalues.

Estimating bounds for $\rho(G)$ is of great interest, and many results have been obtained. Subhi and Powers in [17] proved that for $n \geqslant 3$ the path P_n has the maximum distance spectral radius among trees on n vertices. Stevanović and Ilić in [18] generalized this result, and proved that among trees with fixed maximum degree Δ, the broom graph has maximal distance spectral radius. Denote by $A(n,m)$ $(n \geqslant 2m)$ the tree obtained from the star S_{n-m+1} by attaching a pendent edge to each of certain $m-1$ non-central vertices of S_{n-m+1}. Ilić in [12] proved that the tree $A(n,m)$ minimizes the distance spectral radius among n-vertex trees with matching number m. In [2], Bose et al. showed that among graphs with r pendent vertices, K_n^r is the unique graph with minimal distance spectral radius for $0 \leqslant r \leqslant n-1$ but $r \neq n-2$ and the double star $S(n-3,1)$ is the unique graph with minimal distance spectral radius for $r = n-2$, where K_n^r is a graph obtained by joining r independent vertices to one vertex of the complete graph K_{n-r}. For other results, the readers may refer to [9–22]. In this paper, we characterize the graph with minimum distance spectral radius among trees with fixed number of pendent vertices.

Suppose G is a graph. Let $N(v)$ be the set of vertices adjacent to v in G. The *degree* of v, denoted by $d(v)$, is equal to $|N(v)|$. A vertex of degree one is called a *pendent vertex*. The edge incident with a pendent vertex is known as a *pendent edge*. Denote by $\mathcal{K}(G)$ the number of pendent vertices in G. For $k \geqslant 1$, we say a path $P = wv_1v_2\cdots v_k$ is a pendent path if $d(w) > 2$, $d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2$ (if they exist) and $d(v_k) = 1$. In this case, we say P is *adjacent* to w. Two pendent paths $P = wv_1v_2\cdots v_p$ $(d(v_p) = 1)$ and $Q = vu_1u_2\cdots u_q$ $(d(u_q) = 1)$ are *adjacent* if w and v are the same. The lengths of P and Q are *almost the same* if $|p-q| \leqslant 1$. Set $n_{\geqslant j} = |\{v \in V(G) \mid d(v) \geqslant j\}|$. For $n \geqslant 2$, $m \geqslant 2$, $j \geqslant 1$, let $\mathcal{T}(n,m) = \{T \mid T$ is a tree of order n and $\mathcal{K}(T) = m\}$, $\mathcal{T}(n,m,j) = \{T \in \mathcal{T}(n,m) \mid n_{\geqslant 3} = j\}$ and $\rho_{\min}(\mathcal{T}(n,m)) = \min\{\rho(T) \mid T \in \mathcal{T}(n,m)\}$. It is easy to check that when $n = 2,3,4,5$, there is exactly one tree in $\mathcal{T}(n,m)$, for each possible value of m. The unique tree in $\mathcal{T}(n,2)$ is P_n. So we only consider the case $n \geqslant 6$ and $m \geqslant 3$ in the next sections. For $n \geqslant 6, m \geqslant 3$, denote by $T_{\min}(n,m)$ the tree $T \in \mathcal{T}(n,m,1)$ such that the lengths of any two pendent paths in T are almost the same. Fig. 1 illustrates the graph $T_{\min}(n,m)$ with $n = 8$, $m = 4$.

This paper is organized as follows. In Section 2, we introduce some lemmas that will be used later. In Section 3, we give a graph transformation that decreases the distance spectral radius. Special cases are also studied here. In Section 4, we find the extremal tree that uniquely minimizes the distance spectral radius among trees with fixed number of pendent vertices.

2. Lemmas

Let G be a graph and v_0 be a vertex of G. We denote by $G(v_0,k)$ the graph obtained from G by attaching at the vertex v_0 a pendent path $P = v_0v_1\cdots v_k$ of length k. Let ρ be the distance spectral radius of $G(v_0,k)$ and $x_0, x_1, \ldots, x_k$ be the coordinates of the Perron vector x of $G(v_0,k)$ for vertices v_0, v_1, …, v_k, respectively. Let S be the sum of all coordinates of x. Then we have the following lemma.