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On nonsingularity of block two-by-two matrices

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We derive necessary and sufficient conditions for guaranteeing the nonsingularity of a block two-by-two matrix by making use of the singular value decompositions and the Moore–Penrose pseudoinverses of the matrix blocks. These conditions are complete, and much weaker and simpler than those given by Decker and Keller [D.W. Decker, H.B. Keller, Multiple limit point bifurcation, J. Math. Anal. Appl. 75 (1980) 417–430], and may be more easily examined than those given by Bai [Z.-Z. Bai, Eigenvalue estimates for saddle point matrices of Hermitian and indefinite leading blocks, J. Comput. Appl. Math. 237 (2013) 295–306] from the computational viewpoint. We also derive general formulas for the rank of the block two-by-two matrix by utilizing either the unitarily compressed or the orthogonally projected sub-matrices.

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1. Introduction

We discuss conditions that guarantee the nonsingularity of block two-by-two matrices of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

and derive general formulas for the rank of the matrix M, where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n\times n}$ are complex matrices. It is obvious that, under suitable partitioning, any matrix can be cast in the form (1.1).

Matrices of block two-by-two structures include as special cases the standard and the generalized saddle-point matrices [3,4,31,9] and the skew-Hamiltonian matrices [23,27]. They frequently arise from stability and bifurcation theory of ordinary differential equations [16,17,19,11], order-reduction and sinc discretization of the third-order linear ordinary differential equations [26,7], domain decomposition methods of partial differential equations [10,28,29,4], finite-element discretization and first-order linearization of the two-phase flow problems based on Cahn–Hilliard equation [14,2,13], finite-element discretizations of PDE-constrained optimization problems [21,25,6], real equivalent formulations of complex linear systems [1,6], linear and H_∞ control problems [20,23,24,34,27], matrix completions [15,22,32,33], and so on.

One of the fundamental and important problems is how to examine the nonsingularity or, in general, how to determine the rank of the matrix M. When A is positive semidefinite, D is Hermitian positive semidefinite, and $C = -B^*$ have full rank, i.e., the matrix M is of the generalized saddle-point form, from [12, Theorem 3.4] we know that if

$$\mathrm{null}\big(\mathcal{H}(A)\big) \cap \mathrm{null}\big(B^*\big) = \{0\},$$

then M is nonsingular; and if M is nonsingular, then

$$\mathrm{null}(A) \cap \mathrm{null}\big(B^*\big) = \{0\}.$$

Note that the converses of the above conditions do not hold in general, so they are only either sufficient or necessary. Here the matrix A is said to be positive definite (or semidefinite) if its Hermitian part $\mathcal{H}(A) = \frac{1}{2}(A + A^*)$ is Hermitian positive definite (or semidefinite), with $(\cdot)^*$ and $\mathrm{null}(\cdot)$ denoting the conjugate transpose and the null space of the corresponding matrix, respectively; see [8]. In addition, when $D = 0$ and $m \geqslant n$, [12, Theorem 3.3] showed that if M is nonsingular, then

$$\mathrm{rank}(B) = n \quad \text{and} \quad \mathrm{rank}\begin{pmatrix} A \\ C \end{pmatrix} = n.$$

Note that these conditions are only necessary but sufficient.

In general, there are little results about the nonsingularity of the block two-by-two matrix M. To our knowledge, in [16] Decker and Keller proved the following result about the nonsingularity of the matrix M; see also [17].

Theorem 1.1. *(See [16].) For the block two-by-two matrix M defined in* (1.1)*, the following statements hold true*:

(a) *if A is singular with* $\dim(\mathrm{null}(A)) = n \geqslant 1$, *then M is nonsingular if and only if*
  (a_1) $\dim(\mathrm{range}(B)) = n$,
  (a_2) $\mathrm{range}(A) \cap \mathrm{range}(B) = \{0\}$,
  (a_3) $\dim(\mathrm{range}(C)) = n$, *and*
  (a_4) $\mathrm{null}(A) \cap \mathrm{null}(C) = \{0\}$;
(b) *if A is nonsingular, then M is nonsingular if and only if its Schur complement $S = D - CA^{-1}B$ is nonsingular*;
(c) *if A is singular and* $\dim(\mathrm{null}(A)) > n$, *then M is singular.*