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Barbour path functions and related operator means

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ABSTRACT

We consider Barbour path function $F_x(a,b)=a\cdot\frac{\frac{b}{a}x+\sqrt{\frac{b}{a}}(1-x)}{x+\sqrt{\frac{b}{a}}(1-x)}$ $(0\leqslant x\leqslant 1,\ a,b>0)$ as an approximation of the exponential function (or the geometric mean path) $G_x(a,b)=a^{1-x}b^x$ $(0\leqslant x\leqslant 1,\ a,b>0)$ by a linear fractional function, which interpolates $G_x(a,b)$ at $x=0,\frac{1}{2}$ and 1. If $a=1$ and $b=t$, then both the functions $F_x(1,t)$ and $G_x(1,t)$ are operator monotone in t, parameterized with x.
We also consider the order relation between the integral mean for the Barbour path function and another mean.

1. Introduction

The ideal guitar frets should be located so that the 12 tones of an octave are in the equal temperament, or their respective lengths of the strings on a guitar are proportional to the geometric progression:

$$G_x=2^x\quad\left(x=0,\frac{1}{12},\frac{2}{12},\ldots,\frac{11}{12},1\right).$$

A simple geometric construction giving acceptable approximation was shown by D.P. Strähle. He introduced a linear fractional function:

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$$F_x = \frac{10x+24}{24-7x},$$

which interpolates G_x at $x=0$, $x=1$, and almost so at $x=\frac{1}{2}$, that is, $F_{1/2}=\frac{58}{41}$, approximately equal to $\sqrt{2}=G_{1/2}$.

Generalizing this idea, J.M. Barbour [1] suggested an approximation of the exponential function

$$G_x(a,b)=a^{1-x}b^x \quad (0 \leqslant x \leqslant 1,\ a,b>0),$$

by a linear fractional one:

$$F_x(a,b)=a\cdot\frac{\frac{b}{a}x+\sqrt{\frac{b}{a}}(1-x)}{x+\sqrt{\frac{b}{a}}(1-x)} \quad (0 \leqslant x \leqslant 1,\ a,b>0), \tag{1.1}$$

which interpolates $G_x(a,b)$ at $x=0$, $x=\frac{1}{2}$ and $x=1$.

Notice that $G_x(a,b)$ generalizes to the path of (non-commutative) geometric operator means in the sense of F. Kubo and T. Ando [5]:

$$A\#_x B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^x A^{1/2}$$

for positive operators A and B on a Hilbert space. We will show that $F_x(a,b)$ also generalizes to the operator mean setting; see Corollary 3.4 and the remark after its proof.

As x varies over $[0,1]$, we can regard $F_x(a,b)$ as a path of functions. Replacing the constants in the definition of $F_x(a,b)$ by strictly positive continuous functions leads to the following notion of a Barbour path, named after J.M. Barbour.

Definition 1.1. (See [6].) Let α, β and γ be strictly positive continuous functions on $(0,\infty)$. Then we call the following function *Barbour path* function or simply a *Barbour path*:

$$\varphi_{\alpha,\beta,\gamma}(x)=\frac{\alpha x+\beta(1-x)}{x+\gamma(1-x)} \quad (0 \leqslant x \leqslant 1). \tag{1.2}$$

For example, in (1.1), putting $a=1$ and $b=t$ (or, putting $\alpha=t$ and $\beta=\gamma=\sqrt{t}$ in (1.2)), we have

$$F_x(1,t)\left(=\varphi_{t,\sqrt{t},\sqrt{t}}(x)\right)=\frac{tx+\sqrt{t}(1-x)}{x+\sqrt{t}(1-x)}. \tag{1.3}$$

This function in t is an approximation of $G_x(1,t)=t^x$, which interpolates at the three points $x=0$, $\frac{1}{2}$ and 1.

In this paper, we introduce Barbour path functions as operator monotone functions (or operator means), and consider the integral mean for the Barbour path function, likewise, the integral form of the logarithmic mean, or the integral mean defined by J.I. Fujii and M. Fujii [3]. Furthermore, we discuss the order relation between the integral mean for the Barbour path function and another mean.

2. The order relation between the integral mean for the Barbour path function and another mean

In this section, we consider the integral mean for the Barbour path function (1.1) in Section 1. Now we recall (1.1) as (2.1):

$$F_x(a,b)=a\cdot\frac{\frac{b}{a}x+\sqrt{\frac{b}{a}}(1-x)}{x+\sqrt{\frac{b}{a}}(1-x)} \quad (0 \leqslant x \leqslant 1,\ a,b>0). \tag{2.1}$$

For the most familiar means, the arithmetic mean $A(a,b)$, the geometric mean $G(a,b)$, the harmonic mean $H(a,b)$, the next inequalities are well known: