Linear Algebra and its Applications 439 (2013) 2442–2447

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A sharp upper bound on the signless Laplacian spectral radius of graphs

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ARTICLE INFO

Article history:
Received 5 April 2013
Accepted 14 June 2013
Available online 5 July 2013
Submitted by R. Brualdi

MSC:
05C35
05C50
05C90

Keywords:
Graph
Signless Laplacian matrix
Spectral radius
Upper bound

ABSTRACT

Let G be a simple connected graph of order n with degree sequence $d_1, d_2, \dots, d_n$ in non-increasing order. The signless Laplacian spectral radius $\rho(Q(G))$ of G is the largest eigenvalue of its signless Laplacian matrix $Q(G)$. In this paper, we give a sharp upper bound on the signless Laplacian spectral radius $\rho(Q(G))$ in terms of d_i, which improves and generalizes some known results.

1. Introduction

We only consider simple undirected graphs which have no loops and multiple edges. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. For any two vertices $v_i, v_j \in V$, we write $i \sim j$ if v_i and v_j are adjacent. For any vertex $v_i \in V$, denote the *degree* of v_i by d_i. The *degree sequence* of G is denoted by $\Delta = d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n = \delta$ in non-increasing order.

Let $A(G) = (a_{ij})$ be the adjacency matrix of G and $D(G) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. Then $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G. It is well known that $Q(G)$ is real symmetric, positive semi-definite matrix, which implies that its

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.06.015

eigenvalues are negative real numbers. We may arrange its eigenvalues as $\rho(Q(G)) = \rho_1(Q(G)) \geqslant \rho_2(Q(G)) \geqslant \cdots \geqslant \rho_n(Q(G)) \geqslant 0$. $\rho(Q(G))$ is called the *signless Laplacian spectral radius* of G.

For a simple connected graph G with n vertices, m edges and the degree sequence $\Delta = d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n = \delta$, there are some known upper bounds on the signless Laplacian spectral radius $\rho(Q(G))$ as follows.

In [6], Oliveira et al. proved that

$$\rho\big(Q(G)\big) \leqslant \max_{1\leqslant i\leqslant n}\left\{\frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2}\right\} \tag{1}$$

and

$$\rho\big(Q(G)\big) \leqslant \max_{1\leqslant i\leqslant n}\{d_i + \sqrt{d_i m_i}\}, \tag{2}$$

where $m_i = \frac{1}{d_i}\sum_{j\sim i} d_j$.

In [2,3], Li, Liu et al. obtained that

$$\rho\big(Q(G)\big) \leqslant \frac{\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 8(2m - (n-1)\delta)}}{2}. \tag{3}$$

In 2010, Chen and Wang [1] got that

$$\rho\big(Q(G)\big) \leqslant \frac{\delta - 1 + \sqrt{(\delta - 1)^2 + 8(2m + \Delta^2 - (n-1)\delta)}}{2} \tag{4}$$

and

$$\rho\big(Q(G)\big) \leqslant \frac{2m + \sqrt{m(n^3 - n^2 - 2mn + 4m)}}{n}. \tag{5}$$

Recently, using a similar method to an existing result related to the adjacency spectrum (see Theorem 2.2 in [7]), Yu et al. [8] showed that

$$\rho\big(Q(G)\big) \leqslant \min_{1\leqslant i\leqslant n}\left\{\frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2}\right\}. \tag{6}$$

In this paper, we also give a sharp upper bound on the signless Laplacian spectral radius $\rho(Q(G))$ in terms of d_i, which theoretically improves and generalizes some known results. Moreover, we also determine all extremal graphs which attain this upper bound. Some examples show that this bound is better than above presented results in some cases.

2. Main results

To prove our main results, we firstly need the following Lemma 2.1.

Lemma 2.1. *(See [5].) If A is a nonnegative irreducible $n \times n$ matrix with largest eigenvalue $\rho(A)$ and row-sums $r_1, r_2, \ldots, r_n$, then*

$$\rho(A) \leqslant \max_{1\leqslant i\leqslant n} r_i$$

with equality if and only if the row-sums of A are all equal.

Now we shall provide an improvement of the previous upper bound (6) on the signless Laplacian spectral radius. Remark that the proof of this bound is analogous to an existing result related to the adjacency spectrum (see Theorem 1.7 in [4]).