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Operator norm attainment and inner product spaces

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ABSTRACT

In this paper we prove that a finite dimensional real normed linear space $\mathbb{X}$ is an inner product space iff for any linear operator T on $\mathbb{X}$, T attains its norm at $e_1, e_2 \in S_{\mathbb{X}}$ implies T attains its norm at $span\{e_1, e_2\} \cap S_{\mathbb{X}}$.

1. Introduction

Let $(\mathbb{X}, \|\,\|)$ be a finite dimensional real normed linear space. Let $B_{\mathbb{X}} = \{x \in \mathbb{X}\colon\ \|x\| \leqslant 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X}\colon\ \|x\| = 1\}$ be the unit ball and the unit sphere of the normed linear space $\mathbb{X}$ respectively. Let $\mathbb{L}(\mathbb{X})$ denote the space of all linear operators on $\mathbb{X}$. We use the notion of orthogonality in the sense of Birkhoff–James [3] to prove that a finite dimensional real normed linear space $\mathbb{X}$ is an inner product space if for any linear operator T on $\mathbb{X}$, T attains its norm at $e_1, e_2 \in S_{\mathbb{X}}$ implies T attains its norm at $span\{e_1, e_2\} \cap S_{\mathbb{X}}$.

For any two elements $x, y \in \mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff–James, written as $x \perp_B y$, iff

$$\|x\| \leqslant \|x + \lambda y\| \quad \forall \lambda \in \mathbb{R}.$$

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Likewise for any two elements $T, A \in \mathbb{L}(\mathbb{X})$, T is said to be orthogonal to A, written as $T \perp_B A$, iff

$$\|T\| \leqslant \|T + \lambda A\| \quad \forall \lambda \in \mathbb{R}.$$

It is easy to verify that for $T, A \in \mathbb{L}(\mathbb{X})$ if there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$, then $T \perp_B A$. In a finite dimensional Hilbert space H, Bhatia and Semrl [2] and Paul et al. [5] independently proved that $T \perp_B A$ if and only if there exists $x \in \mathbb{X}$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. Bhatia and Semrl conjectured in their paper that if $\mathbb{X}$ is a finite dimensional normed linear space and $T \perp_B A$ then there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. Li and Schneider [4] gave examples of finite dimensional normed linear spaces $\mathbb{X}$ in which there exist operators $T, A \in \mathbb{L}(\mathbb{X})$ such that $T \perp_B A$ but there exists no $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$, which proved that the conjecture by Bhatia and Semrl is not true. Benítez et al. [1] proved that $\mathbb{X}$ is an inner product space if and only if for $T, A \in \mathbb{L}(\mathbb{X})$ with $T \perp_B A \Leftrightarrow$ there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.

In this paper we prove that if T is a linear operator on a real normed linear space $\mathbb{X}$ such that T attains its norm only at $\pm D$, where D is a connected subset of $S_{\mathbb{X}}$ then $T \perp_B A$ if and only if there exists $x \in S_{\mathbb{X}}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$. Using this result we prove that a finite dimensional real normed linear space $\mathbb{X}$ is an inner product space iff for any linear operator T on $\mathbb{X}$, T attains its norm at $e_1, e_2 \in S_{\mathbb{X}}$ implies T attains its norm at $span\{e_1, e_2\} \cap S_{\mathbb{X}}$.

2. Main results

Theorem 2.1. *Let $\mathbb{X}$ be a finite dimensional real normed linear space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that T attains its norm at only $\pm D$, where D is a connected subset of $S_{\mathbb{X}}$. Then for $A \in \mathbb{L}(\mathbb{X})$ with $T \perp_B A$ there exists $x \in D$ such that $Tx \perp_B Ax$.*

Proof. If possible suppose that there does not exist any $x \in D$ such that $Tx \perp_B Ax$. We now obtain a contradiction in the following three steps to complete the proof of the theorem.

Step 1. In the first step we show that $D = W_1 \cup W_2$ where

$$W_1 = \{x \in D\colon \|Tx + \lambda Ax\| > \|T\| \ \forall \lambda > 0\},$$

$$W_2 = \{x \in D\colon \|Tx + \lambda Ax\| > \|T\| \ \forall \lambda < 0\}.$$

Let $x_0 \in D$ be arbitrary. Since Tx_0 is not orthogonal to Ax_0 in the sense of Birkhoff–James so there exists $\lambda_0 \in \mathbb{R}$ such that $\|Tx_0 + \lambda_0 Ax_0\| < \|Tx_0\| = \|T\|$.

Now either $\lambda_0 > 0$ or $\lambda_0 < 0$. We assume that $\lambda_0 < 0$.

Now, for $\lambda > 0$ $\exists t \in (0, 1)$ such that

$$Tx_0 = t(Tx_0 + \lambda Ax_0) + (1 - t)(Tx_0 + \lambda_0 Ax_0)$$

$$\Rightarrow \quad \|Tx_0\| < t\big\|(Tx_0 + \lambda Ax_0)\big\| + (1 - t)\|Tx_0\|$$

$$\Rightarrow \quad \|Tx_0\| < \big\|(Tx_0 + \lambda Ax_0)\big\|.$$

$$\text{Therefore } \|Tx_0 + \lambda Ax_0\| > \|Tx_0\| = \|T\| \quad \forall \lambda > 0.$$

If we assume that $\lambda_0 > 0$ then we can similarly show that

$$\|Tx_0 + \lambda Ax_0\| > \|Tx_0\| = \|T\| \quad \forall \lambda < 0.$$

Thus for $x \in D$ either $\|Tx + \lambda Ax\| > \|T\| \ \forall \lambda > 0$ or $\|Tx + \lambda Ax\| > \|T\| \ \forall \lambda < 0$ and so $D = W_1 \cup W_2$.

Step 2. We now prove that $W_1 \neq \phi$ and $W_2 \neq \phi$.