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Polynomial numerical hulls of some normal matrices

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ABSTRACT

In this note, polynomial numerical hulls of matrices of the form $A_1 \oplus iA_2$, where A_1 and A_2 are Hermitian, are characterized.

1. Introduction

Let $A \in M_n$. The polynomial numerical hull of order k of A was first introduced by O. Nevanlinna [8] as

$$V^k(A) := \{z \in \mathbb{C}: \forall p \in P_k,\ |p(z)| \leqslant \|p(A)\|\},$$

where P_k is the set of complex polynomials of degree at most k, and $\|\cdot\|$ is the spectral norm. Also, some generalizations of polynomial numerical hull were defined by Salemi [9]. Remember that the joint numerical range of matrices $A_1, \dots, A_m \in M_n$ is defined as follows:

$$W(A_1, \dots, A_m) = \{(x^* A_1 x, \dots, x^* A_m x)^T \in \mathbb{C}^m: x \in \mathbb{C}^n,\ x^* x = 1\}.$$

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Greenbaum [7] showed that when A is normal,

$$V^k(A) = \{z \in \mathbb{C}\colon (z, z^2, \ldots, z^k) \in W(A, A^2, \ldots, A^k)\}.$$

Besides, Dash [4] showed that joint numerical range of any m-tuple of commuting normal matrices is the convex hull of joint spectrum of it. So

$$V^k(A) = \{z \in \mathbb{C}\colon (z, z^2, \ldots, z^k) \in \operatorname{conv}(\sigma(A, A^2, \ldots, A^k))\}. \tag{1}$$

This formula plays a key role in the arguments presented in this paper. For matrices $A = A_1 \oplus iA_2$ where A_1, A_2 are Hermitian; $V^2(A)$ was computed by Davis and Salemi [6]. Also, in [5] it is proved that the polynomial numerical hull of order 4 or higher of any matrix of this type is the spectrum. We determined the polynomial numerical hull of order 3 for 4-by-4 matrices of this form [2], and presented a formula [1] for matrices with size greater than 6-by-6, but there is not any complete characterization for matrices of remained sizes. So, we concentrate on 5-by-5 and 6-by-6 ones to reach to the goal of this paper.

For the rest of the paper we shall fix the following notations

$$\begin{aligned} aS &= \{as\colon s \in S\},\\ S + a &= \{s + a\colon s \in S\},\\ {}_{a,b}r_{c,d,e} &= \frac{cde + ab(c + d - e)}{cde(b - a)},\\ {}_{a,b}s_{c,d,e} &= \frac{cde(b - a)}{cde + ab(c + d - e)}, \end{aligned}$$

where $a, b, c, d, e \in \mathbb{C}$ and $S \subset \mathbb{C}$.

In addition, throughout this paper, any interval whose lower bound is strictly greater than its upper will be considered as an empty set.

Below, we state some properties of the polynomial numerical hull of $A \in M_n$.

Lemma 1.1. *(See [6–8].) Let $A \in M_n$, $k \in \mathbb{N}$ and $U \in M_n$ be a unitary matrix. Then:*

(a) *$V^k(A)$ is a compact set.*
(b) *$\sigma(A) \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq V^1(A) = W(A)$, for all $k \geqslant 1$.*
(c) *If m is the degree of the minimal polynomial of A, then $V^k(A) = \sigma(A)$ for all $k \geqslant m$.*
(d) *$V^k(\alpha A + \beta I) = \alpha V^k(A) + \beta$ for all α and β in the complex plane $\mathbb{C}$.*
(e) *Let A be a normal matrix and $\sigma(A) \subset \mathbb{R} \cup i\mathbb{R}$, then $V^k(A) \subset \mathbb{R} \cup i\mathbb{R}$.*
(f) *Let $A = A^*$. Then $V^2(A) = \sigma(A)$.*
(g) *Let $A' = A \oplus [\alpha]$, where $[\alpha]$ denotes a 1×1 matrix with entry $\alpha \in V^k(A)$, then $V^k(A') = V^k(A)$.*
(h) *$V^k(\bar{A}) = \{\bar{z}\colon z \in V^k(A)\}$, where $\bar{z}$ is complex conjugate of z.*
(i) *$V^k(U^*AU) = V^k(A)$.*

Theorem 1.2. *(See [2].) Let $A = \operatorname{diag}(\alpha, -\beta, i\gamma, -i\theta)$ where α, β, γ and θ are positive numbers. Then:*

(a) *If $\alpha = \beta$ or $\gamma = \theta$, then $V^3(A) = \sigma(A) \cup (\{\frac{\alpha\beta(\theta-\gamma)i+\theta\gamma(\beta-\alpha)}{\alpha\beta+\theta\gamma}\} \cap W(A))$.*
(b) *If $\alpha \neq \beta$ and $\gamma \neq \theta$, then $V^3(A) = \sigma(A)$.*

Theorem 1.3. *(See [2].) Let A be of the form $A = A_1 \oplus iA_2$, where $A_1^* = A_1$, $A_2^* = A_2$ and A_2 be a semi-definite matrix. Then $V^3(A) = \sigma(A)$.*

Theorem 1.4. *(See [1].) Let $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ has distinct eigenvalues. Then the following results emerge:*