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Matrix commutators: their asymptotic metric properties and relation to approximate joint diagonalization

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ABSTRACT

We analyze the properties of the norm of the commutator of two Hermitian matrices, showing that asymptotically it behaves like a metric, and establish its relation to joint approximate diagonalization of matrices, showing that almost-commuting matrices are almost jointly diagonalizable, and vice versa. We show an application of our results in the field of 3D shape analysis.

1. Introduction

The study of almost-commuting matrices has attracted interest in the theoretical community since the 1950s,[1] mainly motivated by problems arising in quantum mechanics, where it was important to establish whether two almost-commuting matrices are close to matrices that exactly commute [1,22,24,16,13,10,17,12].

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[1] Prof. Terry Loring has drawn our attention to the fact that almost-commutativity can be traced back to the work of John von Neumann [20] in the late 1920s.

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On the other hand, the well-known fact that commuting matrices are jointly diagonalizable relates almost-commuting matrices to numerical methods for simultaneous diagonalization of matrices. Recently, such methods have been used in signal processing [7,6,8], machine learning [9], and computer graphics [15]. Kovnatsky et al. [15] used approximate joint diagonalization of Laplacians in order to construct coupled Fourier bases on manifolds representing 3D shapes, which allowed performing such operations as editing multiple shapes together and finding correspondence between them. Furthermore, the joint diagonalizability of Laplacians was used as a criterion of the similarity of the underlying 3D shapes [15].

In light of these applications, it is important to understand the behavior of almost-commuting or almost jointly diagonalizable matrices. In this paper, we show that *almost-commuting self-adjoint matrices are almost jointly diagonalizable by a unitary matrix, and vice versa*, in a sense that will be explained later. Besides theoretical interest, this result has practical applications given the recent use of simultaneous approximate diagonalization in various applications. Since the joint diagonalization procedure is computationally expensive, the easily computable norm of the commutator can be used instead; our result justifies this use.

Furthermore, we study distances between matrices using the norm of their commutator. We show that while not a metric, such a construction is a (pseudo-)metric asymptotically for sufficiently large matrices with a "good" distribution. Such a distance is zero between jointly diagonalizable matrices, allowing to employ it for estimating the intrinsic similarity of manifolds by comparing their Laplacians [15].

2. Background and definitions

Let A, B be two $n \times n$ complex matrices. We denote by

$$\|A\|_F = \Big(\sum_{ij} |a_{ij}|^2\Big)^{1/2} = \big(\mathrm{tr}(A^*A)\big)^{1/2};$$

$$\|A\|_2 = \max_{x\in\mathbb{C}^n:\, \|x\|_2=1} \|Ax\|_2 = \big(\lambda_{\max}(A^*A)\big)^{1/2},$$

the *Frobenius* and the *operator* norm (induced by the Euclidean vector norm) of A, respectively. Here A^* is the adjoint (conjugate transpose) of A.

We say that A, B are *jointly diagonalizable* if there exists a unitary matrix U such that $U^*AU = \Lambda_A$ and $U^*BU = \Lambda_B$ are diagonal. In general, two matrices A, B are not necessarily jointly diagonalizable, however, we can approximately diagonalize them by minimizing

$$\min_U J(A, B, U) \quad \text{s.t.} \quad U^*U = I$$

where

$$J(A, B, U) = \mathrm{off}\big(U^*AU\big) + \mathrm{off}\big(U^*BU\big),$$

and $\mathrm{off}(A) = \sum_{i\neq j} |a_{ij}|^2$ is the sum of the squared absolute values of the off-diagonal elements. In the following, we denote $J(A, B) = \min_{U^*U=I} J(A, B, U)$. Numerically, this optimization problem can be solved by a Jacobi-type iteration, referred to as the JADE algorithm [5,8].

We say that A and B *commute* if $AB = BA$, and call $[A, B] = AB - BA$ their *commutator* (in the following, we use both notations). It is well known that commuting self-adjoint matrices are jointly diagonalizable [14], which can be expressed as $\|[A, B]\|_F = 0$ iff $J(A, B) = 0$. The case of *almost-commuting* matrices has been studied extensively in [1,22,24,16,13,10,17,12], showing that almost-commuting matrices are close to commuting matrices, e.g., in the following sense:

Theorem 2.1 *(Lin, 1997). There exists a function $\epsilon(\delta)$ satisfying $\lim_{\delta\to 0} \epsilon(\delta) = 0$ with the following property: If A, B are two self-adjoint $n \times n$ matrices satisfying $\|A\|_2, \|B\|_2 \leqslant 1$, and $\|AB - BA\|_2 \leqslant \delta$, then there exists a pair A', B' of* commuting matrices *satisfying $\|A - A'\|_2 \leqslant \epsilon(\delta)$ and $\|B - B'\|_2 \leqslant \epsilon(\delta)$.*