Linear Algebra and its Applications 439 (2013) 1700–1715

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# A symbolic treatment of Riordan arrays ☆

José Agapito [a], Ângela Mestre [a], Pasquale Petrullo [b], Maria M. Torres [a,*]

[a] Centro de Estruturas Lineares e Combinatórias, Universidade de Lisboa, Portugal
[b] Universitá degli studi della Basilicata, Potenza, Italy

ARTICLE INFO

*Article history:*
Received 11 September 2012
Accepted 8 May 2013
Available online 14 June 2013
Submitted by R. Brualdi

*Keywords:*
Riordan arrays
Umbral calculus
Abel's identity

ABSTRACT

We approach Riordan arrays and their generalizations via umbral symbolic methods. This new approach allows us to derive fundamental aspects of the theory of Riordan arrays as immediate consequences of the umbral version of the classical Abel's identity for polynomials. In particular, we obtain a novel non-recursive formula for Riordan arrays and derive, from this new formula, some known recurrences and a new recurrence relation for Riordan arrays.



## 1. Introduction

A Riordan array is an infinite lower triangular matrix $R = (r_{n,k})_{n,k\in\mathbb{N}}$ *ordinarily* described by a pair of generating functions $(d(t), h(t))$ such that $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$, and $r_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ is the operator which gives the $n$th coefficient in the series development of a generating function. If $d(t) = \sum_{n\geqslant 0} d_n \frac{t^n}{n!}$, $h(t) = \sum_{n\geqslant 1} h_n \frac{t^n}{n!}$, and $r_{n,k} = n![t^n]d(t)\frac{h(t)^k}{k!}$, then $R$ is called *exponential* Riordan array. More generally, given a sequence $(c_n)$ of nonzero numbers and taking $d(t) = \sum_{n\geqslant 0} d_n \frac{t^n}{c_n}$, $h(t) = \sum_{n\geqslant 1} h_n \frac{t^n}{c_n}$, and $r_{n,k} = c_n[t^n]d(t)\frac{h(t)^k}{c_k}$, $R$ is called a *generalized* Riordan array with respect to $(c_n)$ [38]. Thus, for example, the Riordan array whose entries are the binomial numbers $\binom{n}{k}$ can be described by the pair $(\frac{1}{1-t}, \frac{t}{1-t})$ ($c_n = 1$: ordinary presentation) or by $(e^t, t)$

☆ This work was done within the activities of the Centro de Estruturas Lineares e Combinatórias (University of Lisbon, Portugal) and the Dipartimento di Matematica e Informatica (Università degli Studi della Basilicata, Italy). The first author was partially supported by the Portuguese Science and Technology Foundation (FCT) through the program Ciência 2008 and grant PTDC/MAT/099880/2008; the second author was supported by the FCT grant SFRH/BPD/48223/2008 and the fourth author was partially supported by the FCT and FEDER/POCI 2010.

* Corresponding author.
*E-mail addresses:* jagruiz@cii.fc.ul.pt (J. Agapito), mestre@cii.fc.ul.pt (Â. Mestre), p.petrullo@gmail.com (P. Petrullo), mmtorres@fc.ul.pt (M.M. Torres).

0024-3795/$ – see front matter 
http://dx.doi.org/10.1016/j.laa.2013.05.007

($c_n = n!$: exponential presentation). Riordan arrays form a group under matrix multiplication. The literature about Riordan arrays is vast and still growing and the applications cover a wide range of subjects, such as enumerative combinatorics, combinatorial sums, recurrence relations and computer science, among other topics [5,13–18,31–33,35]. Formally, ordinary Riordan arrays are a formulation of the 1-umbral calculus, whereas exponential Riordan arrays are a formulation of the $n!$-umbral calculus. In fact, the classical umbral calculus, as given in [27], consists of a systematic study of a certain class of univariate polynomial families (*Sheffer sequences*) by employing linear operators on polynomials. An extension of the classical umbral calculus to infinitely many variables can be found in [3]. Sheffer sequences, provided with a particular non-commutative multiplication (*umbral composition*), form a group which is isomorphic to the Riordan group [12].

The purpose of this paper is to give a promising new symbolic treatment of Riordan arrays, based on a renewed approach to umbral calculus initiated by Rota and Taylor in [30]. This renewed approach makes no use of operator theory. The symbolic techniques developed from this new approach have been fruitfully applied to a wide range of topics [7–11,20,22–25,28,29,36,37]. In particular, Di Nardo, Niederhausen and Senato [7] gave a representation of exponential Riordan arrays as a result of their symbolic handling of Sheffer polynomials. Definition 3.1 given later on this paper is a normalized version of their formulation. A treatment of Riordan arrays under the more general context of *recursive* matrices using the more established approach to umbral calculus [27] can be found in [2,4].

A key tool in our symbolic approach to Riordan arrays is the umbral version (formula (4)) of the following binomial Abel identity for polynomials

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (y+ka)^{n-k} x(x-ka)^{k-1}. \tag{1}$$

The umbral analogue of (1) was first stated in [9] (without a proof) and later obtained from an umbral version of the Lagrange inversion formula in [7]. An elementary proof of the umbral version of Abel's identity has been recently given by Petrullo [23,24].

The main contribution of this paper is to show that well-known properties of all Riordan arrays are easily derived from the umbral version of Abel's identity. This is a nontrivial aspect from a theoretical point of view. It is not immediate by means of the traditional methods for dealing with Riordan arrays. It is worth mentioning that, in the classical context of generating functions, Sprugnoli [34] had already used Riordan arrays to obtain several combinatorial formulas that generalize the classical Abel's identity. Therefore, our point of view is conceptually different from Sprugnoli's approach.

The paper is organized as follows. Section 2 recalls the basics of Rota and Taylor's classical umbral calculus [30]. Along the way, two equivalent umbral versions of Abel's identity for polynomials and an umbral version of the classical Lagrange inversion formula are given. Section 3 states the fundamental aspects of the theory of Riordan arrays (of exponential type) in umbral terms. Also, connections with Sheffer sequences are recast in the new umbral symbolic setting. Furthermore, some recurrence relations of Riordan arrays are stated. Section 4 tests our umbral approach to Riordan arrays on some concrete classical examples, providing both umbral and traditional formulas. Section 5 extends the discussion given in Section 3 to $\omega$-Riordan arrays, where $\omega$ is any umbra with nonzero moments. In this generalized context, we obtain an important umbral formula (Theorem 5.3) as a direct consequence of the umbral Abel identity (4). One of our main results is a novel non-recursive formula (Theorem 5.4), which is a direct application of Theorem 5.3. Known recurrence relations and a new recurrence formula for $\omega$-Riordan arrays (Theorems 5.5, 5.6 and 5.7) are derived from Theorem 5.4.

## 2. Preliminaries

This section starts with a brief review of the essentials of the classical umbral calculus that underlies much of this paper. We recall the intimate relation that exists between the umbral version of Abel's identity (1) and the Lagrange inversion formula. This connection is not clear at first sight from the traditional point of view using generating functions (see for instance [6] for the common view of these two classical formulas). The umbral versions of the binomial Abel identity and of the Lagrange inversion formula play an important role in our symbolic approach to Riordan arrays (see Sections 3 and 5).