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Products of projections and positive operators

M. Laura Arias [a,1], Gustavo Corach [a,b,*,2], M. Celeste Gonzalez [a,c,3]

[a] *Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, Piso 3, 1083 Buenos Aires, Argentina*
[b] *Dpto. de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina*
[c] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*

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ABSTRACT

This article is devoted to the study of the set $\mathcal{T}$ of all products PA with P an orthogonal projection and A a positive (semidefinite) operator. We describe this set and study optimal factorizations. We also relate this factorization with the notion of compatibility and explore the polar decomposition of the operators in $\mathcal{T}$.

1. Introduction

Given two classes of operators $\mathcal{M}$ and $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space), a problem which naturally arises is that of characterizing the set $\mathcal{M}\cdot\mathcal{B}$ of all products AB, $A \in \mathcal{M}$, $B \in \mathcal{B}$. These problems are as old as matrix theory and they form now an interesting part of factorization theory for matrices and operators. In 1958 Chandler Davis [8, Theorem 6.3] proved that, if $\mathcal{I}$ denotes the set of Hermitian involutions (i.e., $T = T^* = T^{-1}$) then $\mathcal{I}\cdot\mathcal{I}$ coincides with all unitaries T such that T is similar to T^{-1}. H. Radjavi and J.P. Williams [21] proved later that $\mathcal{I}\cdot\mathcal{L}^h$, where $\mathcal{L}^h$ denotes the set of Hermitian operators on $\mathcal{H}$, is the set of all $T \in \mathcal{L}(\mathcal{H})$ such that T is unitarily equivalent to T^{-1}. Their paper

* Corresponding author at: IAM-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.
E-mail addresses: lauraarias@conicet.gov.ar (M.L. Arias), gcorach@fi.uba.ar (G. Corach), celeste.gonzalez@conicet.gov.ar (M.C. Gonzalez).
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also contains a characterization of $\mathcal{P}\cdot\mathcal{P}$ due to T. Crimmins and a characterization of $\mathcal{P}\cdot\mathcal{L}^h$ (here, $\mathcal{P}$ denotes the set of all orthogonal projectors of $\mathcal{L}(\mathcal{H})$). Other characterizations of $\mathcal{P}\cdot\mathcal{P}$ have been found by S. Nelson and M. Neumann [17], A. Arias and S. Gudder [1], T. Oikhberg [18] and the second author and A. Maestripieri [6]. In a series of papers, J.R. Holub [14–16] (see also Fujii and Furuta [12]) studied, as an approach to general Wiener–Hopf or Toeplitz operators, some properties of the class $\mathcal{P}\cdot\mathcal{G}^+=\{PA\colon\ P\in\mathcal{P}$ and $A\in\mathcal{L}^+$ is invertible$\}$, where $\mathcal{L}^+$ denotes the cone of positive semidefinite operators in $\mathcal{L}(\mathcal{H})$. They observed that the set $\mathcal{Q}$ of oblique (i.e., not necessarily orthogonal) projections in $\mathcal{L}(\mathcal{H})$ is contained in $\mathcal{P}\cdot\mathcal{G}^+$.

In this paper, we characterize operators in $\mathcal{T}:=\mathcal{P}\cdot\mathcal{L}^+$. We extend several results on $\mathcal{P}\cdot\mathcal{P}$ and Holub's theorem that $\mathcal{Q}$ is contained in $\mathcal{P}\cdot\mathcal{G}^+$. It should be noticed that $\mathcal{Q}$ is not contained in $\mathcal{P}\cdot\mathcal{P}$, but it is contained in $(\mathcal{P}\cdot\mathcal{P})^\dagger$, the set of all Moore–Penrose inverses of products PQ, $P,Q\in\mathcal{P}$. This is an old result by Penrose [20] and Greville [13] which has been extended to the infinite dimensional case in [5] and [4]. The paper [21] by H. Radjavi and J. Williams and the survey [25] by P.Y. Wu contain many characterizations of classes of the type $\mathcal{M}\cdot\mathcal{B}$.

One of the main features of the class $\mathcal{P}\cdot\mathcal{L}^+$ is that their elements admit a particular polar decomposition where the partial isometry is an orthogonal projection. In fact, for $T\in\mathcal{T}$, any factorization $T=PA$, with $P\in\mathcal{P}$ and $A\in\mathcal{L}^+$ provides one such polar decomposition. Among all these expressions, we find one (the optimal factorization) with some relevant minimal properties. The main characterization of $\mathcal{T}$ is based on a result of Z. Sebestyén [22]. We include a proof, which is completely different from the original one, because it illustrates how the classical majorization theorem of R.G. Douglas [10,11] can be used to provide special solutions of some operator equations. In fact, if $T\in\mathcal{T}$ and P is the orthogonal projection onto the closure of the image of T, then the positive solutions of the equation $PX=T$ play a natural role in this paper.

The contents of the paper are the following. Section 2 contains notations and the statements of some theorems by Crimmins [11, Theorem 2.2], Douglas [10, Theorem 1] and Sebestyén [22]. We include a proof of the last one based on Douglas' theorem. Section 3 is devoted to several properties of the set $\mathcal{T}$ and different characterizations of its elements. Just to mention two of them, $T\in\mathcal{L}(\mathcal{H})$ belongs to $\mathcal{T}$ if and only if there exists $\lambda\geqslant 0$ such that $(T^*T)^2\leqslant\lambda T^*T^2$ (Theorem 3.2). If $R(T)$ is closed then $T\in\mathcal{T}$ if and only if $R(T)\dotplus N(T)=\mathcal{H}$ and $TP\in\mathcal{L}^+$, where $P=P_{R(T)}$ (Theorem 3.3). A formula for the oblique projection onto $R(T)$ with nullspace $N(T)$ is exhibited at Section 4, where a particular factorization of $T\in\mathcal{T}$ is shown to have several optimal properties. For instance, if $T\in\mathcal{T}$ then there exist $P_T\in\mathcal{P}$ and $A_T\in\mathcal{L}^+$ such that $T=P_TA_T$ and $P_T\leqslant P$ and $A_T\leqslant A$ for all $P\in\mathcal{P}$, $A\in\mathcal{L}^+$ such that $T=PA$. The last result of Section 4 is the characterization of the fiber of $T\in\mathcal{T}$ by the map $(P,A)\to PA$, i.e., we find all pairs $(P,A)\in\mathcal{P}\times\mathcal{L}^+$ such that $PA=T$. In Section 5 we relate the different factorizations of $T\in\mathcal{T}$ with the notions of compatibility and quasi-compatibility between positive operators and closed subspaces. It turns out that, if $T\in\mathcal{T}$ and $T=PA$ for some $P=P_{\mathcal{S}}\in\mathcal{P}$ and $A\in\mathcal{L}^+$, then the pair $(A,\mathcal{S})$ is compatible if and only if $\mathcal{H}=\overline{R(T)}\dotplus N(T)$. The last section studies some properties of the standard polar decomposition of $T\in\mathcal{T}$.

2. Preliminaries

Throughout $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{K}$ denote separable complex Hilbert spaces. By $\mathcal{L}(\mathcal{H},\mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $\mathcal{L}(\mathcal{H},\mathcal{H})$ is abbreviated by $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}(H)^+$ we denote the cone of positive (semidefinite) operators of $\mathcal{L}(\mathcal{H})$ i.e., $T\in\mathcal{L}(\mathcal{H})^+$ if and only if $\langle Tx,x\rangle\geqslant 0$ for all $x\in\mathcal{H}$. Furthermore, $\mathcal{G}(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$ and $\mathcal{CR}(\mathcal{H})$ the set of closed range operators on $\mathcal{H}$. When no confusion can arise, we omit the Hilbert space and we write it simply $\mathcal{L}^+,\mathcal{G}$ and $\mathcal{CR}$ respectively. Moreover, we denote $\mathcal{G}^+=\mathcal{G}\cap\mathcal{L}^+$. Given $T\in\mathcal{L}(\mathcal{H},\mathcal{K})$, $R(T)$ denotes the range or image of T, $N(T)$ the nullspace of T, T^* the adjoint of T and $T^\dagger$ the Moore–Penrose inverse of T. Recall that $T^\dagger\in\mathcal{L}(\mathcal{K},\mathcal{H})$ if and only if $R(T)$ is closed. We shall denote by $\mathcal{Q}=\{Q\in\mathcal{L}(\mathcal{H})\colon\ Q=Q^2\}$ and $\mathcal{P}=\{P\in\mathcal{Q}\colon\ P=P^*\}$. Moreover, fixed a closed subspace $\mathcal{S}$, $P_{\mathcal{S}}$ stands for the orthogonal projection onto $\mathcal{S}$. In the sequel we denote by $\mathcal{S}\dotplus\mathcal{W}$ the direct sum of the subspaces $\mathcal{S}$ and $\mathcal{W}$. In particular, if $\mathcal{S}\subseteq\mathcal{W}^\perp$ we denote $\mathcal{S}\oplus\mathcal{W}$.

We end this section by stating three important results that we will frequently use along this article.