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The spectrum of the Hilbert space valued second derivative with general self-adjoint boundary conditions ☆

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ABSTRACT

We consider a large class of self-adjoint elliptic problems associated with the second derivative acting on a space of vector-valued functions. We present and survey several results that can be obtained by means of two different approaches to the study of the associated eigenvalues problems. The first, more general one allows to replace a secular equation (which is well known in some special cases) by an abstract rank condition. The second one, though available in general, seems to apply particularly well to a specific boundary condition, the sometimes dubbed *anti-Kirchhoff condition* in the literature, that arises in the theory of differential operators on graphs; it also permits to discuss interesting and more direct connections between the spectrum of the differential operator and some graph theoretical quantities, in particular some results on the symmetry of the spectrum in either case.

1. Introduction

Differential equations on networks have a long history, starting probably in 1847 with Kirchhoff's electrical circuit equations using the potential mesh rule and an incident currency law [32]. Ever since,

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this kind of equations has been re-discovered several time in different contexts: quantum chemistry, theoretical biology, operator theory, machine learning, quantum chaos.... In most of these applications the relevant objects are not functions solely defined in the nodes of the network: They are rather functions defined on each edge of the network. For this reason, one is forced to replace systems of ordinary differential equations by systems of partial ones – the specific nature of the partial differential equation clearly depends on the physical process one wants to describe. However, most of the above mentioned problems are associated with second order differential operators with suitable boundary conditions – this is also the specific setting we will focus on in this note. Our aim here is both to survey some known results on the spectral problem for the second derivative and to show how said results can be obtained following certain recurrent theoretical patterns. In particular, we are going to present two possible approaches to this field and show how they can also be exploited to show some results that seem to be unknown.

For specific second order differential operators acting on functions defined on the edges – which, say, describe a potential – both conditions originally proposed by Kirchhoff were naturally replaced by the continuity potential condition and an incident flow condition. Nowadays, they have been variously generalized. Moreover, it is often interesting to allow for networks with infinitely many edges. Both issues suggest that we rather consider abstract interacting problems associated with differential operators acting on spaces of vector-valued functions (typically, of Bochner type) with general coupled boundary conditions. The problem of determining all possible self-adjoint realizations of a differential operator is quite common in mathematical physics, as it is always conceivable to prepare a corresponding quantum system in such a way that the associated Hamiltonian is one such realization.

Also because of the interest of mathematical physicists in operators theory on networks, it is no surprise that in recent years most researchers active in the field of differential equations on networks have focused on spectral problems. In particular, we would like to point out how two different methods can be effectively employed to study direct and inverse spectral problems on graphs. The first one is the transformation of a differential equation on a graph into a vector-valued differential equation on an interval: this easy idea makes most computations easier, since we can drop geometric consideration and treat different graphs (on the same number of edges) simply as different self-adjoint realizations of the same operator, and is reviewed in Section 2. If however the boundary conditions do encode much information and are therefore naturally associated with an underlying graph, then it is often convenient to develop a formalism that takes them into account: this can be done by using the so-called *adjacency calculus*, which we recall in Section 3. Finally, having related in the previous section the Laplacian with two different classes of boundary conditions with a relevant matrix of graph theory, we show in Section 4 that several results of spectral graph theory have a direct counterpart in the spectral analysis of Laplacians and allow us to fully exploit discrete symmetries of underlying graphs.

Relying upon known results for differential operators on domains, in [33] V. Kostrykin and R. Schrader have proposed a natural representation of self-adjoint boundary conditions for 1-dimensional systems, i.e., for Laplace operators on a Bochner-type Hilbert space $L^2(0,1;\ell^2(E))$. If E is the edge set of a graph, then such systems are usually called "networks" or "quantum graphs" in the literature. That approach has the drawbacks that it is poorly fitted for a variational setting, and that the boundary conditions do not determine their representations uniquely. Both issues can be avoided making use of an alternative parametrization proposed by P. Kuchment in [34]. After Kuchment's parametrization has become popular, some authors have observed that the same representation of boundary conditions had already been studied in [29]. However, we would like to point out that this kind of boundary conditions goes back at least to Hölder, cf. [31, §3], [4,5] and the references therein.

Both parametrizations are equivalent, in the sense that each Kuchment's boundary condition can be represented using Kostrykin–Schrader's formalism, and vice versa. Moreover, if E is finite, i.e., if one considers a differential operator on finitely many edges, then it has been shown in [34] that self-adjoint boundary conditions are actually exhausted by either parametrization. Kuchment's parametrization can be reduced to the choice of a closed subspace Y of $\ell^2(E)$ and of a bounded linear self-adjoint R on $\ell^2(E)$; see Section 2 for details. To the best of our knowledge, not much is known about spectral properties of general self-adjoint realizations, beside the resolvent formula of