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The classification of Leonard triples of Racah type

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ABSTRACT

Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard triple* on V we mean an ordered triple of linear transformations A, A^*, A^ε in $\mathrm{End}(V)$ such that for each $B \in \{A, A^*, A^\varepsilon\}$ there exists a basis for V with respect to which the matrix representing B is diagonal and the matrices representing the other two linear transformations are irreducible tridiagonal. In this paper we define a family of Leonard triples said to have Racah type and classify them up to isomorphism. Moreover, we show that each of them satisfies the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. As an application, we construct all Leonard triples that have Racah type from the universal enveloping algebra $U(sl_2)$.

1. Introduction

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field of characteristic zero.

This paper is about the classification of Leonard triples that have Racah type. Before going into details, we recall some terms. A square matrix X is said to be *upper* (resp. *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal. X is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero.

The notion of a Leonard pair was introduced by Terwilliger in [10].

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Definition 1.1. (See [10, Definition 1.1].) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

Terwilliger classified the Leonard pairs up to isomorphism in [12]. By that classification, the isomorphism classes of Leonard pairs fall naturally into thirteen families: q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, affine q-Krawtchouk, quantum q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito and orphan.

The notion of a Leonard triple was introduced by Curtin in [3].

Definition 1.2. (See [3, Definition 1.2].) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard triple* on V, we mean an ordered triple of linear transformations $A : V \to V$, $A^* : V \to V$ and $A^\varepsilon : V \to V$ that satisfy the conditions (i)–(iii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ε are irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A and A^ε are irreducible tridiagonal.
(iii) There exists a basis for V with respect to which the matrix representing A^ε is diagonal and the matrices representing A and A^* are irreducible tridiagonal.

For any Leonard triple, any two of the three form a Leonard pair. We say these Leonard pairs are *associated* with the Leonard triple.

The classification of Leonard triples is also very important. Curtin classified up to isomorphism the modular Leonard triples in [3]. Brown classified up to isomorphism the totally bipartite and the totally almost bipartite Leonard triples of Bannai/Ito type in [2]. Huang classified up to isomorphism the Leonard triples of q-Racah type in [6]. In this paper motivated by [6], we define a family of Leonard triples said to be Racah type and classify them up to isomorphism.

The paper is organized as follows. In Sections 2–5 we recall some background concerning Leonard pairs and Leonard triples. In Section 6 we consider a certain sequence in order to study a family of Leonard systems said to have Racah type. In Section 7 we define a family of Leonard systems said to have Racah type and discuss some related concepts. In Sections 8, 9 we define a family of Leonard pairs said to have Racah type. For a given Leonard pair (A, A^*) that has Racah type, we show that there exists a unique $A^\varepsilon \in \mathrm{End}(V)$ such that A, A^*, A^ε satisfy the $\mathbb{Z}_3$-symmetric Askey–Wilson relations, and give the necessary and sufficient conditions for (A, A^*, A^ε) to be a Leonard triple. In Sections 10, 11 we define a family of Leonard triples said to have Racah type. We classify up to isomorphism the Leonard triples that have Racah type, and show that each of them satisfies the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. In Section 12, as an application, we construct all Leonard triples that have Racah type from the universal enveloping algebra $U(sl_2)$.

2. Leonard pairs and Leonard systems

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract object called a Leonard system.

For an integer $d \geqslant 0$, let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. Its rows are indexed by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$. For the rest of the paper, V will denote a vector