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Positive semidefinite zero forcing

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ABSTRACT

The positive semidefinite zero forcing number $\mathrm{Z}_+(G)$ of a graph G was introduced in Barioli et al. (2010) [4]. We establish a variety of properties of $\mathrm{Z}_+(G)$: Any vertex of G can be in a minimum positive semidefinite zero forcing set (this is not true for standard zero forcing). The graph parameters $\mathrm{tw}(G)$ (tree-width), $\mathrm{Z}_+(G)$, and $Z(G)$ (standard zero forcing number) all satisfy the Graph Complement Conjecture (see Barioli et al. (2012) [3]). Graphs having extreme values of the positive semidefinite zero forcing number are characterized. The effect of various graph operations on positive semidefinite zero forcing number and connections with other graph parameters are studied.

1. Introduction

Every graph discussed is simple (no loops or multiple edges), undirected, and has a finite nonempty vertex set. In a graph G where some vertices S are colored black and the remaining vertices are colored white, the *positive semidefinite color change rule* is: If $W_1, \ldots, W_k$ are the sets of vertices of the k components of $G - S$ (note that it is possible that $k = 1$), $w \in W_i$, $u \in S$, and w is

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the only white neighbor of u in the subgraph of G induced by $W_i \cup S$, then change the color of w to black; in this case, we say u *forces* w and write $u \to w$. Given an initial set B of black vertices, the *derived set* of B is the set of black vertices that results from applying the positive semidefinite color change rule until no more changes are possible. A *positive semidefinite zero forcing set* is an initial set B of vertices such that the derived set of B is all the vertices of G. The *positive semidefinite zero forcing number* of a graph G, denoted $\mathrm{Z}_+(G)$, is the minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$. The positive semidefinite zero forcing number is a variant of the (*standard*) *zero forcing number* $\mathrm{Z}(G)$, which uses the same definition with a different color change rule: If u is black and w is the only white neighbor of u, then change the color of w to black. The (standard) zero forcing number was introduced in [1] as an upper bound for maximum nullity, and the positive semidefinite zero forcing number was introduced in [4] as an upper bound for positive semidefinite maximum nullity.

Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\}\colon a_{ij} \neq 0 \text{ and } i \neq j\}$. The *maximum positive semidefinite nullity* of G is

$$\mathrm{M}_+(G) = \max\bigl\{\operatorname{null} A\colon\ A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\bigr\}$$

and *minimum positive semidefinite rank* of G is

$$\mathrm{mr}_+(G) = \min\bigl\{\operatorname{rank} A\colon\ A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\bigr\}.$$

The (standard) maximum nullity $\mathrm{M}(G)$ and (standard) minimum rank $\mathrm{mr}(G)$ use the same definitions omitting the requirement of positive semidefiniteness. It is clear that $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$. In [4] it was shown that for every graph

$$\mathrm{M}_+(G) \leqslant \mathrm{Z}_+(G).$$

It was also shown there that

$$OS(G) + \mathrm{Z}_+(G) = |G|$$

where $OS(G)$ is a graph parameter defined in [14], and in fact shown that the complement of an OS-set is a positive semidefinite zero forcing set and the complement of a positive semidefinite zero forcing set is an OS-set. The reader is referred to [14] for the definition of OS-set and $OS(G)$.

We establish a variety of properties of $\mathrm{Z}_+(G)$. In Section 2 connections between zero forcing sets and OS-sets are applied to show that every vertex of G is in some minimum positive semidefinite zero forcing set (this is not true for standard zero forcing). It is also shown there that $\mathrm{T}(G) \leqslant \mathrm{Z}_+(G)$ where $\mathrm{T}(G)$ is the tree cover number of G, and cut-vertex reduction formulas for $\mathrm{T}(G)$ and $\mathrm{Z}_+(G)$ are established. In Section 3 it is shown that the graph parameters $\mathrm{tw}(G)$ (tree-width), $\mathrm{Z}_+(G)$, and $\mathrm{Z}(G)$ (standard zero forcing number) all satisfy the Graph Complement Conjecture (see [3]). Graphs having extreme values of the positive semidefinite zero forcing number are characterized in Section 4. The effect of various graph operations on positive semidefinite zero forcing number and connections with other graph parameters are studied in Section 5.

There are a few more graph terms that we need to define. The subgraph $G[W]$ of $G = (V, E)$ *induced* by $W \subseteq V$ is the subgraph with vertex set W and edge set $\{\{i, j\} \in E\colon\ i, j \in W\}$; $G - W$ is used to denote $G[V \setminus W]$. The graph $G - \{v\}$ is also denoted by $G - v$. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets from V that are not in E. The *union* of $G_i = (V_i, E_i)$ is $\bigcup_{i=1}^h G_i = (\bigcup_{i=1}^h V_i, \bigcup_{i=1}^h E_i)$. The *intersection* of $G_i = (V_i, E_i)$ is $\bigcap_{i=1}^h G_i = (\bigcap_{i=1}^h V_i, \bigcap_{i=1}^h E_i)$ (provided the intersection of the vertices is nonempty). The *degree* of vertex v in graph G, $\deg_G v$, is the number of neighbors of v. A graph is *chordal* if it has no induced cycle of length 4 or more; clearly any induced subgraph of a chordal graph is chordal.

2. Tree cover number, positive semidefinite zero forcing number, and maximum positive semidefinite nullity

The *tree cover number* of a graph G, denoted $\mathrm{T}(G)$, is defined as the minimum number of vertex disjoint trees occurring as induced subgraphs of G that cover all of the vertices of G, and was