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On the Fiedler value of large planar graphs

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ABSTRACT

The Fiedler value λ_2, also known as algebraic connectivity, is the second smallest Laplacian eigenvalue of a graph. We study the maximum Fiedler value among all planar graphs G with n vertices, denoted by $\lambda_{2\,\max}$, and we show the bounds $2+\Theta(\frac{1}{n^2}) \leqslant \lambda_{2\,\max} \leqslant 2+O(\frac{1}{n})$. We also provide bounds on the maximum Fiedler value for the following classes of planar graphs: Bipartite planar graphs, bipartite planar graphs with minimum vertex-degree 3, and outerplanar graphs. Furthermore, we derive almost tight bounds on $\lambda_{2\,\max}$ for two more classes of graphs, those of bounded genus and K_h-minor-free graphs.

1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V=\{v_1,\ldots,v_n\}$. The *Laplacian matrix* $L(G)$ is the $n\times n$ matrix with entries

$$\ell_{i,j}=\begin{cases} deg(v_i) & \text{if } i=j,\\ -1 & \text{if } i\neq j \text{ and } v_iv_j\in E,\\ 0 & \text{if } i\neq j \text{ and } v_iv_j\notin E.\end{cases}$$

Let the eigenvalues of $L(G)$ be $0=\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_n$. The second smallest eigenvalue λ_2, or $\lambda_2(G)$, is called the *Fiedler value* or *algebraic connectivity* [7] of G. It is related to a number of graph invariants and it plays a special role in many problems in Physics and Chemistry, where spectral techniques can be applied [1,7,15,16]. Another classical problem for which the techniques introduced

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Table 1
The bounds on $\lambda_{2\max}$ obtained for each class of graphs studied.

Planar graphs	$2+\Theta(\frac{1}{n^2}) \leqslant \lambda_{2\max} \leqslant 2+O(\frac{1}{n})$
Bipartite planar graphs, $\delta=3$	$1+\Theta(\frac{1}{n^2}) \leqslant \lambda_{2\max} \leqslant 1+O(\frac{1}{n^{1/3}})$
Bipartite planar graphs, n large	$\lambda_{2\max}=2$
Outerplanar graphs	$1+\Theta(\frac{1}{n^2}) \leqslant \lambda_{2\max} \leqslant 1+O(\frac{1}{n})$
Graphs of bounded genus g	$2+\Theta(\frac{1}{n^2}) \leqslant \lambda_{2\max} \leqslant 2+O(\frac{1}{\sqrt{n}})$
K_h-minor-free graphs, $4 \leqslant h \leqslant 9$	$h-2 \leqslant \lambda_{2\max} \leqslant h-2+O(\frac{1}{\sqrt{n}})$
K_h-minor-free graphs, h large	$\lambda_{2\max} \leqslant \alpha h\sqrt{\log(h)}+O(\frac{\alpha h^{5/2}\sqrt{\log(h)}}{\sqrt{n}})$ for $\alpha=0.319\ldots+o(1)$

in [7] have revealed to be very successful is graph partitioning [6]. The Fiedler value has also been proved to be related to the size of separators, as well as to the quality of geometric embeddings of the graph [19,20].

A number of results have been obtained for λ_2, for which we refer the interested reader to the surveys [1,16]. As for recent works, the authors of [4] make use of flows and the choice of an appropriate metric for proving bounds on λ_2. Similar techniques are used in [12] to study higher eigenvalues of graphs of bounded genus. Another work devoted to upper bounds on the algebraic connectivity is [8].

The main goal of the present work is to study the maximum of $\lambda_2(G)$ over all planar graphs G with n vertices, which will be denoted as $\lambda_{2\max}$. The bound $\lambda_{2\max} < 6$ follows easily, since for any graph $G=(V,E)$ with n vertices $\lambda_2(G) \leqslant \frac{2|E|}{n-1}$ [7] and if G is planar then $|E| \leqslant 3n-6$. Molitierno [17] proved that $\lambda_{2\max} \leqslant 4$, with exactly two planar graphs attaining this bound: the complete graph with four vertices, K_4, and the octahedral graph $K_{2,2,2}$. It is known that λ_2 is much smaller for some planar graph classes. In particular, trees have $\lambda_2 \leqslant 1$, with the bound achieved only for $K_{1,n-1}$ [14]. Moreover, Spielman and Teng [20] proved that for the class of bounded-degree planar graphs with n vertices, $\lambda_{2\max}$ tends towards zero when n tends towards infinity.

We also study $\lambda_{2\max}$ for bipartite planar graphs, outerplanar graphs, graphs of bounded genus and K_h-minor-free graphs. Table 1 summarizes our results. Some of them improve our previous results presented in [3].

For all upper bounds on $\lambda_{2\max}$ we make use of the following Embedding Lemma, which makes clear the relation between geometric embeddings of graphs and the Fiedler value. It is a direct consequence of the so-called Courant–Fischer principle and can be found in [15,20].

Lemma 1.1 *(Embedding Lemma). Let $G=(V,E)$ be a graph. Then λ_2, the Fiedler value of G, is given by*

$$\lambda_2 = \min \frac{\sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i=1}^{n} \|\vec{v}_i\|^2}$$

where the minimum is taken over all non-zero vectors $\{\vec{v}_1,\ldots,\vec{v}_n\} \subset \mathbb{R}^n$ such that $\sum_{i=1}^{n} \vec{v}_i = \vec{0}$.

We will make use of the Embedding Lemma 1.1 in two ways. Before introducing them, let us state a result by Spielman and Teng for planar graphs.

Theorem 1.2. *(See Spielman and Teng [20].) Let G be a planar graph with n vertices and maximum degree Δ. Then, the Fiedler value of G is at most $\frac{8\Delta}{n}$.*