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On matrices over an arbitrary semiring and their generalized inverses

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ABSTRACT

In this paper, we consider matrices with entries from a semiring $\mathbf{S}$. We first discuss some generalized inverses of rectangular and square matrices. We establish necessary and sufficient conditions for the existence of the Moore–Penrose inverse of a regular matrix. For an $m \times n$ matrix A, an $n \times m$ matrix P and a square matrix Q of order m, we present necessary and sufficient conditions for the existence of the group inverse of QAP with the additional property that $P(QAP)^{\#}Q$ is a $\{1,2\}$ inverse of A. The matrix product used here is the usual matrix multiplication. The result provides a method for generating elements in the set of $\{1,2\}$ inverses of an $m \times n$ matrix A starting from an initial $\{1\}$ inverse of A. We also establish a criterion for the existence of the group inverse of a regular square matrix. We then consider a semiring structure $(\mathbf{M}_{m\times n}(\mathbf{S}), +, \circ)$ made up of $m \times n$ matrices with the addition defined entry-wise and the multiplication defined as in the case of the Hadamard product of complex matrices. In the semiring $(\mathbf{M}_{m\times n}(\mathbf{S}), +, \circ)$, we present criteria for the existence of the Drazin inverse and the Moore–Penrose inverse of an $m \times n$ matrix. When $\mathbf{S}$ is commutative, we show that the Hadamard product preserves the Hermitian property, and provide a Schur-type product theorem for the product $A \circ (CC^*)$ of a positive semidefinite $n \times n$ matrix A and an $n \times n$ matrix C.

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1. Introduction

The concept of the generalized inverse seems to have been first mentioned in print by Fredholm [4], where a particular generalized inverse of an integral operator was given. The class of all generalized inverses was characterized in 1912 by Hurwitz [11]. The algebraic nature of generalized inverses of matrices was established in the works of Drazin [3], Moore [16], Penrose [21] and others. The theory of generalized inverses of real and complex matrices is a well-developed subject and the results of this theory and its applications can be found in many well-known monographs; see for instance [13] and [26]. Some researchers studied generalized inverses of matrices, such as the Moore–Penrose inverse, the group inverse and the Drazin inverse, in more general algebraic settings like commutative rings [2], arbitrary rings [19,24,25], arbitrary field [9] and [20] and idempotent semirings [18]. This prompted us to investigate some of the generalized inverses for matrices over an arbitrary semiring. The concept of semirings was introduced by H.S. Vandiver in 1935, and since then it has been studied by many authors (see, e.g., [6,7]). Semirings constitute a fairly natural generalization of rings. The theory of matrices over semirings has important applications in optimization theory and graph theory; see [1] and [5].

A semiring consists of a nonempty set $\mathbf{S}$ with two binary operations on $\mathbf{S}$, addition $(+)$ and multiplication $(\cdot)$, such that the following conditions hold:

(1) $(\mathbf{S},+)$ is an Abelian monoid with an identity element denoted by 0 (unless otherwise stated).
(2) $(\mathbf{S},\cdot)$ is a monoid with an identity element denoted by 1 (unless otherwise stated).
(3) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in\mathbf{S}$.
(4) $a\cdot 0=0\cdot a=0$ for all $a\in\mathbf{S}$.
(5) $1\neq 0$.

We use, unless otherwise stated, the symbol $\mathbf{S}$ to denote both the set and the semiring structure, and usually write ab instead of $a\cdot b$ for all $a,b\in\mathbf{S}$. If $a\in\mathbf{S}$ is invertible, we write $a^{-1}a^{-1}$ as a^{-2}. It should be noted that every ring with a unity is a semiring. A semiring $\mathbf{S}$ is called *commutative* if $ab=ba$ for all $a,b\in\mathbf{S}$. Examples of commutative semirings include the two-element Boolean algebra $\mathbf{B}=\{0,1\}$ and the fuzzy algebra $\mathbf{F}=\{t\colon\ 0\leqslant t\leqslant 1\}$ with the binary operations $a+b=\max\{a,b\}$ and $ab=\min\{a,b\}$ in both examples, and the nonnegative integers and nonnegative real numbers with standard operations of addition and multiplication. A simple example of a non-commutative semiring can be constructed as follows: Let $\mathbf{S}$ be a semiring which is additively non-idempotent, that is, $1+1\neq 1$. Then the set of all 2×2 matrices over $\mathbf{S}$ with the usual matrix addition and multiplication (see Definition 1.1) is a non-commutative semiring. The non-commuting property could be seen by observing that $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\neq\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$.

Throughout this paper the symbol $\mathbf{S}$ is taken to be a semiring, and m and n are taken to be positive integers. We denote the set of positive integers by $\mathbb{N}$, and the set $\{1,\ldots,n\}$ is denoted by $\langle n\rangle$. We denote by $\mathbf{M}_{m\times n}(\mathbf{S})$ the set of all $m\times n$ matrices with entries from $\mathbf{S}$. The set $\mathbf{M}_{m\times m}(\mathbf{S})$ is simply written as $\mathbf{M}_m(\mathbf{S})$. We define $\mathbf{M}(\mathbf{S};m,n)$ by

$$\mathbf{M}(\mathbf{S};m,n)=\mathbf{M}_{m\times n}(\mathbf{S})\cup\mathbf{M}_{n\times m}(\mathbf{S})\cup\mathbf{M}_m(\mathbf{S})\cup\mathbf{M}_n(\mathbf{S}).\tag{1.1}$$

The transpose of $A\in\mathbf{M}_{m\times n}(\mathbf{S})$ is denoted by A^t. If $A=(a_{ij})\in\mathbf{M}_{m\times n}(\mathbf{S})$, we also write a_{ij} as $(A)_{ij}$. The trace of $A=(a_{ij})\in\mathbf{M}_n(\mathbf{S})$, denoted by $\operatorname{tr}(A)$, is $\sum_{i=1}^n a_{ii}$. A matrix $D\in\mathbf{M}_n(\mathbf{S})$ is called diagonal if all of its off-diagonal entries are the additive identity in $\mathbf{S}$. If a diagonal matrix $D\in\mathbf{M}_n(\mathbf{S})$ satisfies $(D)_{ii}=d_i$ for all $i\in\langle n\rangle$, we write D as $\operatorname{diag}(d_1,\ldots,d_n)$.

Definitions 1.1 and 1.2 introduce semiring structures on $\mathbf{M}_n(\mathbf{S})$ and $\mathbf{M}_{m\times n}(\mathbf{S})$, respectively. The multiplication operation is a profound difference between the two structures.

Definition 1.1. Let $A=(a_{ij})\in\mathbf{M}_{m\times n}(\mathbf{S})$, $B=(b_{ij})\in\mathbf{M}_{m\times n}(\mathbf{S})$ and $C=(c_{ij})\in\mathbf{M}_{n\times m}(\mathbf{S})$. We define the addition $A+B$ and multiplication $A\cdot C$ by