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Unital affine semigroups

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ABSTRACT

Affine semigroups are convex sets on which there exists an associative binary operation which is affine separately in either variable. They were introduced by Cohen and Collins in 1959. We look at examples of affine semigroups which are of interest to matrix and operator theory and we prove some new results on the extreme points and the absorbing elements of certain types of affine semigroups. Most notably we improve a result of Wendel that every invertible element in a compact affine semigroup is extreme by extending this result to linearly bounded affine semigroups.

1. Introduction

In [8], Cohen and Collins introduced the concept of an affine semigroup. They were motivated by the example of the set of all Borel measures on a topological group. These measures form a convex set which is also a semigroup under the natural convolution product. There were several more papers on this subject, most motivated by the theory of measures on topological groups; a good summary of this work can be found in [6]. We further extend this work by considering examples from matrix analysis and operator theory and we improve some key results in this area.

The next section includes an introduction to the necessary background. In Section 3, we broaden the focus to look at affine semigroups that are subsets of matrix algebras. In Section 4, we examine the relation of an affine semigroup to its extreme points. We show that all invertible elements of a linearly bounded affine semigroup are extreme points thereby improving a result of Wendel. We discuss the invertible extreme points of bounded affine semigroups that arise as the unit balls of induced matrix

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norms. Section 5 deals with absorbing elements. We review some cases where an absorbing element exists and we investigate the algebraic structure of the absorbing element when it exists. In Section 6 we explore an alternative perspective on bounded affine semigroups: we look at affine semigroups as sets of linear operators preserving families of convex sets. Using this perspective we find another sufficient condition for an affine semigroup to have an absorbing element.

2. Preliminaries

We begin by reviewing the definitions from [8]. Let S and T be convex sets. Recall that a mapping $f : S \to T$ is said to be *affine* if $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in S$ for $0 \leqslant \lambda \leqslant 1$.

Definition 2.1. (See [8].) An *Affine Semigroup* is a convex set K together with an associative binary operation $\cdot$ with the property that for any $y \in K$ the left multiplication operator $L_y(x) = yx$ and the right multiplication operator $R_y(x) = xy$ are both affine maps from K to K.

Thus an affine semigroup is a set that satisfies the conditions of a convex set and semigroup simultaneously. We will be particularly interested in unital affine semigroups which are subsets of algebras where the binary operation is the multiplication operation of the algebra; these are subsets of an algebra which are closed under convex combinations and multiplication:

(1) $\lambda x + (1-\lambda)y \in X$ for all $x, y \in X$ and $\lambda \in [0,1]$.
(2) $xy \in X$ for all $x, y \in X$.
(3) There is an element $1 \in X$ such that $1x = x1 = x$ for all $x \in X$.

Given a set X, recall the convex hull of X, $conv(X)$, is the smallest convex set containing all the elements of X. If $X = \{x_i\}_{i=1}^n$, then $conv(X) = \{\sum_{i=1}^n \lambda_i x_i \mid \lambda_i \geqslant 0$ for all i, $\sum_{i=1}^n \lambda_i = 1\}$. If x is a point in a convex set K, and for all $\lambda \in [0,1]$, $x = \lambda y + (1-\lambda)z$, $y, z \in K$, implies $y = z = x$, then we say x is an *extreme point* of K. Alternatively, this means that the set $K \setminus \{x\}$ is convex.

If the underlying algebra is a Banach algebra we can further restrict ourselves to certain classes of affine semigroups; for example bounded affine semigroups or compact affine semigroups. There are certain cases where the underlying algebra is a real or complex algebra without an obvious norm or topology, but we would still like some notion of boundedness here as well. Berg and Nikodym formulated the following weaker form of boundedness in their study of convex sets in linear spaces without topology [2].

Definition 2.2. Let K be a convex subset of a real or complex vector space V. We say that K is *linearly bounded* if for all $u, v \in K$ with $v \neq 0$ the set $\{t \in \mathbb{R}\colon\ u + tv \in K\}$ is a bounded interval.

We note that all bounded convex sets in a normed vector space are linearly bounded. In finite dimensional normed vector spaces, all linearly bounded sets are bounded. However in any infinite dimensional normed space there exist linearly bounded convex sets which are not bounded [10].

2.1. Finitely generated affine semigroups and polynomials

An example of a bounded affine semigroup is given by the set of polynomials $\mathcal{P} = \{p(x) = \sum_{i=1}^n a_i x^i \mid a_i \geqslant 0$ and $p(1) = 1\}$. The set is bounded under the norm $\|p\| = \sup_{\{z\colon |z| \leqslant 1\}} |p(z)|$; in particular $\|p\| \leqslant 1$ for all $p \in \mathcal{P}$. This is the affine semigroup generated by x.

Another class of affine semigroups arise from the theory of positive polynomials. If M is a subset of $\mathbb{R}$, then the set of all real valued polynomials which are positive (or alternatively which are non-negative) on M forms an affine semigroup.

The following theorem of Bernstein describes one such affine semigroup; it will be very useful to us later on. We have rephrased this result in the language of affine semigroups; in fact this compact restatement of Bernstein's theorem can be viewed as one application of affine semigroups.