Linear Algebra and its Applications 439 (2013) 2158–2166

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Families of Artin–Schreier curves with Cartier–Manin matrix of constant rank

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ARTICLE INFO

Article history:
Received 2 February 2012
Accepted 11 June 2013
Available online 2 July 2013
Submitted by R. Guralnick

MSC:
15A04
15B33
11G20
14H40

Keywords:
Cartier operator
Cartier–Manin matrix
Artin–Schreier curve
Jacobian
a-Number

ABSTRACT

Let k be an algebraically closed field of characteristic $p > 0$. Every Artin–Schreier k-curve X has an equation of the form $y^p - y = f(x)$ for some $f(x) \in k(x)$ such that p does not divide the least common multiple L of the orders of the poles of $f(x)$. Under the condition that $p \equiv 1 \bmod L$, Zhu proved that the Newton polygon of the L-function of X is determined by the Hodge polygon of $f(x)$. In particular, the Newton polygon depends only on the orders of the poles of $f(x)$ and not on the location of the poles or otherwise on the coefficients of $f(x)$. In this paper, we prove an analogous result about the a-number of the p-torsion group scheme of the Jacobian of X, providing the first non-trivial examples of families of Jacobians with constant a-number. Equivalently, we consider the semi-linear Cartier operator on the sheaf of regular 1-forms of X and provide the first non-trivial examples of families of curves whose Cartier–Manin matrix has constant rank.

1. Introduction

Suppose k is an algebraically closed field of characteristic $p > 0$ and X is an Artin–Schreier k-curve, namely a smooth projective connected k-curve which is a $\mathbb{Z}/p$-Galois cover of the projective line. Studying the p-power torsion of the Jacobian of X is simultaneously feasible and challenging. For example, zeta functions of Artin–Schreier curves over finite fields are analyzed in [13,14,16,19]. Newton polygons of Artin–Schreier curves are the focus of the papers [1–3,9,20].

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.06.012

Every Artin–Schreier k-curve X has an equation of the form $y^p - y = f(x)$ for some non-constant rational function $f(x) \in k(x)$ such that p does not divide the order of any of the poles of $f(x)$. The genus of X depends only on the orders of the poles of $f(x)$. Let $m+1$ denote the number of poles of $f(x)$ and let $d_0, \ldots, d_m$ denote the orders of the poles. By the Riemann–Hurwitz formula, the genus of X is $g_X = D(p-1)/2$ where $D = \sum_{j=0}^{m}(d_j+1) - 2$. By definition, the p-rank of the Jacobian $\mathrm{Jac}(X)$ of X is the dimension s_X of $\mathrm{Hom}_{\mathbb{F}_p}(\mu_p, \mathrm{Jac}(X)[p])$ where μ_p denotes the kernel of Frobenius morphism F on the multiplicative group scheme $\mathbb{G}_m$. The p-rank also equals the length of the slope 0 portion of the Newton polygon. For an Artin–Schreier curve X, the p-rank s_X equals $m(p-1)$ by the Deuring–Shafarevich formula, and thus depends only on the number of poles of $f(x)$.

In most cases, the Newton polygon of X is not determined by the orders of the poles of $f(x)$. One exception was found by Zhu: let L denote the least common multiple of the orders of the poles of $f(x)$; under the condition that $p \equiv 1 \bmod L$, the Newton polygon of X, shrunk by the factor $p-1$ in the horizontal and vertical direction, equals the Hodge polygon of $f(x)$ [25, Corollary 1.3], see Remark 3.1. In particular, this means that the Newton polygon depends only on the orders of the poles of $f(x)$ and not on the location of the poles or otherwise on the coefficients of $f(x)$. In this paper, we prove an analogous result about the a-number of the Jacobian $\mathrm{Jac}(X)$ or, equivalently, about the rank of the Cartier–Manin matrix of X.

The a-number is an invariant of the p-torsion group scheme $\mathrm{Jac}(X)[p]$. Specifically, if α_p denotes the kernel of Frobenius on the additive group $\mathbb{G}_a$, then the a-number of (the Jacobian of) X is $a_X = \dim_k \mathrm{Hom}(\alpha_p, \mathrm{Jac}(X)[p])$. It equals the dimension of the intersection of $\mathrm{Ker}(F)$ and $\mathrm{Ker}(V)$ on the Dieudonné module of $\mathrm{Jac}(X)[p]$, where V is the Verschiebung morphism. The a-number and the Newton polygon place constraints upon each other, but do not determine each other, see e.g. [11,12].

The a-number is the co-rank of the Cartier–Manin matrix, which is the matrix for the modified Cartier operator on the sheaf of regular 1-forms of X. The modified Cartier operator is the $1/p$-linear map $\mathcal{C}: H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ taking exact 1-forms to zero and satisfying $\mathcal{C}(f^{p-1}df) = df$. In other words, the a-number equals the dimension of the kernel of $\mathcal{C}$ on $H^0(X, \Omega_X^1)$.

In this paper, under the condition $p \equiv 1 \bmod L$, we prove that the a-number of X depends only on the orders of poles of $f(x)$ and not on the location of the poles or otherwise on the coefficients of $f(x)$ (see Section 3.6).

Theorem 1.1. *Let X be an Artin–Schreier curve with equation $y^p - y = f(x)$, with $f(x) \in k(x)$. Suppose $f(x)$ has $m+1$ poles, with orders $d_0, \ldots, d_m$, and let $L = \mathrm{LCM}(d_0, \ldots, d_m)$. If $p \equiv 1 \bmod L$, then the a-number of X is*

$$a_X = \sum_{j=0}^{m} a_j, \quad \text{where } a_j = \begin{cases} (p-1)d_j/4 & \text{if } d_j \text{ even,} \\ (p-1)(d_j-1)(d_j+1)/4d_j & \text{if } d_j \text{ odd.} \end{cases}$$

To our knowledge, Theorem 1.1 provides the first non-trivial examples of families of Jacobians with constant a-number when $p \geqslant 3$. When $p = 2$, the main result of [7] is that the Ekedahl–Oort type (and a-number) of an Artin–Schreier curve depend only on the orders of the poles of $f(x)$. For arbitrary p, it is easy to construct families of Jacobians with $a_X = 0$ (ordinary) or $a_X = 1$ (almost ordinary) and a family of Jacobians with $a_X = 2$ is constructed in [10, Corollary 4].

For fixed p, the families in Theorem 1.1 occur for every genus g which is a multiple of $(p-1)/2$. The a-number of each curve in the family is roughly half of the genus. Using [17, Theorem 1.1(2)], the dimension of the family can be computed to be $\sum_{i=0}^{m}(d_j+1) - 3 = 2g/(p-1) - 1$.

Other results about a-numbers of curves can be found in [6,8]. We end the paper with some open questions motivated from this work.

2. Background

2.1. Artin–Schreier curves

Let k be an algebraically closed field of characteristic $p > 0$. A *curve* in this paper is a smooth projective connected k-curve. An *Artin–Schreier curve* is a curve X which admits a $\mathbb{Z}/p$-Galois cover